Generating functions for orthogonal polynomials

of (anti)symmetric multivariate sine functions of three variables

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Main message: Four families of orthogonal polynomials have rational generating functions with the *same* denominator; the four cases differ only by an explicitly computed numerator.

1. Sine/cosine functions

Let $n \in \mathbb{N}$, $\lambda = (\lambda_1, \dots, \lambda_n)$ and $x = (x_1, \dots, x_n)$. Define

$$(S_\lambda(x))_{ij} = \sin(\pi \lambda_i x_j), \qquad (C_\lambda(x))_{ij} = \cos(\pi \lambda_i x_j).$$

The symmetric and antisymmetric functions are

$$\sin^-_\lambda(x) = \det S_\lambda(x), \quad \sin^+_\lambda(x) = \operatorname{perm} S_\lambda(x),$$
$$\cos^-_\lambda(x) = \det C_\lambda(x), \quad \cos^+_\lambda(x) = \operatorname{perm} C_\lambda(x).$$

For the polynomial variables we use

$$X_1 = \cos^+_{(1,0,0)}, \qquad X_2 = \cos^+_{(1,1,0)}, \qquad X_3 = \cos^+_{(1,1,1)}.$$

2. Polynomial families

Set

$$\rho^-_{\mathrm{II}} = (n, n-1, \dots, 1), \qquad \rho^+_{\mathrm{II}} = (1, 1, \dots, 1),$$
$$\rho^-_{\mathrm{IV}} = \left(n - \tfrac{1}{2}, n - \tfrac{3}{2}, \dots, \tfrac{1}{2}\right), \qquad \rho^+_{\mathrm{IV}} = \left(\tfrac{1}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}\right).$$

For $a \in \{\mathrm{II}, \mathrm{IV}\}$, $b \in \{+, -\}$ and

$$P^+ = \{(k_1, \dots, k_n) \in \mathbb{Z}^n : k_1 \ge k_2 \ge \dots \ge k_n \ge 0\},$$

the polynomials are defined by

$$P^{a,b}_{\boldsymbol{k}}(X_1(x), \dots, X_n(x)) = \frac{\sin^b_{\boldsymbol{k}+\rho^b_a}(x)}{\sin^b_{\rho^b_a}(x)}.$$

In this work we specialize to $n = 3$.

Examples: $\mathrm{II}, -$ family

$$P^{\mathrm{II},-}_{(0,0,0)} = 1, \qquad P^{\mathrm{II},-}_{(1,1,0)} = 2X_2 + 2,$$
$$P^{\mathrm{II},-}_{(2,0,0)} = X_1^2 - 2X_2 - 3,$$
$$P^{\mathrm{II},-}_{(2,1,1)} = \tfrac{4}{3}X_1X_3 + X_1^2 - 2X_2 - 2,$$
$$P^{\mathrm{II},-}_{(2,2,0)} = 4X_2^2 - \tfrac{4}{3}X_1X_3 - 2X_1^2 + 10X_2 + 6.$$

3. Generating function

For $n = 3$, write the generating function as

$$G_{a,b}(P, Q, R) = \sum_{(p,q,r)\in P^+} P^{a,b}_{(p,q,r)}(x, y, z)\, P^p Q^q R^r.$$

The computed form is the rational expression

$$\boxed{G_{a,b}(P, Q, R) = \frac{\mathcal{N}_{a,b}(P, Q, R; x, y, z)}{S(P)S(Q)S(R)}}$$

where the denominator is universal:

$$S(U) = 3U^6 - 3xU^5 + 3(2y+3)U^4$$
$$- 2(3x + 2z)U^3 + 3(2y+3)U^2 - 3xU + 3.$$

trigonometric definitions → coefficient generating series → explicit rational formula

4. Numerator structure

Each case has the form

$$\mathcal{N}_{a,b} = \sum c^{a,b}_{pqr}(x, y, z)\, P^p Q^q R^r,$$

with all nonzero coefficients computed explicitly. The support sizes are small enough for direct implementation:

Family	p range	q range	r range
$\mathcal{N}_{\mathrm{II},-}$	$0 \dots 5$	$0 \dots 5$	$0 \dots 4$
$\mathcal{N}_{\mathrm{II},+}$	$0 \dots 4$	$0 \dots 4$	$0 \dots 4$
$\mathcal{N}_{\mathrm{IV},-}$	$0 \dots 5$	$0 \dots 5$	$0 \dots 5$
$\mathcal{N}_{\mathrm{IV},+}$	$0 \dots 5$	$0 \dots 5$	$0 \dots 5$

Representative entries for $\mathcal{N}_{\mathrm{II},-}$:

$$c_{0,2,2} = 9(3x^2 + 4xz + 6y + 9),$$
$$c_{1,2,2} = -9(2y+3)(3x+4z),$$
$$c_{1,3,2} = 108(y+1)^2,$$
$$c_{2,2,2} = 6(9x^2 + 18xz + 9y + 8z^2 + 9),$$
$$c_{4,2,2} = 9(3x^2 + 4xz + 3).$$

5. Coefficient extraction

The rational form replaces recurrence-only computation by coefficient extraction:

$$P^{a,b}_{(p,q,r)}(x, y, z) = [P^p Q^q R^r]\, \frac{\mathcal{N}_{a,b}}{S(P)S(Q)S(R)}.$$

This is useful for:

- **fast evaluation** of many polynomials from one generating function;
- **symbolic exploration** of parity and symmetry relations;
- **implementation**: numerator tables plus one denominator routine;
- **verification**: the first coefficients recover known low-degree polynomials.

Implementation viewpoint

A minimal computational representation is:

1. store the four sparse numerator coefficient tables;
2. define the universal $S(U)$ once;
3. expand the rational expression to the needed total degree;
4. read off $P^{a,b}_{(p,q,r)}$ as the coefficient of $P^p Q^q R^r$.

The main saving is conceptual as well as computational: **one denominator, four numerators, all ordered triples**.

Conclusion

For the three-variable case, the generating functions of the four polynomial families

$$P^{\mathrm{II},-}_{\boldsymbol{k}}, \quad P^{\mathrm{II},+}_{\boldsymbol{k}}, \quad P^{\mathrm{IV},-}_{\boldsymbol{k}}, \quad P^{\mathrm{IV},+}_{\boldsymbol{k}}$$

were computed explicitly. The common denominator $S(P)S(Q)S(R)$ makes the structure transparent, while the sparse numerator coefficients provide an effective route to direct calculation.

Note. The full coefficient lists are intentionally not printed here: they are long, but mechanical. The central result is the rational form and the shared denominator; the coefficient tables will appear in the proceedings.

Reference

[1] A. Brus, J. Hrivnák and L. Motlochová, *Discrete Transforms and Orthogonal Polynomials of (Anti)symmetric Multivariate Sine Functions*, Entropy 20(12), 2018, 938.

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