Poisson structures and Banach Lie algebroids

Tomasz Goliński

Faculty of Mathematics, University of Białystok

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Plan of the talk

1. Finite dimensional picture
2. Banach Lie algebra and its related Poisson structure
3. Banach Lie algebroid

Finite dimensional picture: Lie algebras

$\mathfrak{g}$ — Lie algebra
$\mathfrak{g}^*$ — dual of the Lie algebra

Lie–Poisson bracket

$$f, g \in C^\infty(\mathfrak{g}^*), \qquad \mu \in \mathfrak{g}^*$$

$$\{f, g\}(\mu) = \langle \mu; [f'(\mu), g'(\mu)] \rangle$$

$$f'(\mu) \in (\mathfrak{g}^*)^* \cong \mathfrak{g}$$

Lie algebra $\leftrightsquigarrow$ (linear) Poisson structure on its dual

DEFINITION

Lie algebroid E — a vector bundle over M with a Lie bracket

$$[\,\cdot\,,\,\cdot\,] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$$

and a smooth bundle morphism covering identity (called *anchor*)

$$a : E \to TM$$

such that

1. $[X, fY] = a(X)f \cdot Y + f[X, Y]$ for all $X, Y \in \Gamma(E), f \in C^\infty(M)$,
2. $(\Gamma(E), [\,\cdot\,,\,\cdot\,])$ is a Lie algebra.

E^* — dual bundle of E

Two distinguished families of local functions on E^*:

- pull-backs of local functions $f \in C^\infty_{loc}(M)$ on the base

$$f \circ \pi^*$$

- fiber-wise linear functions given as pairing with local sections $X \in \Gamma_{loc}(E)$:

$$\lambda_X(\rho) = \left\langle \rho \,;\, X_{\pi^*(\rho)} \right\rangle$$

Poisson bracket is given by:

$$\{f \circ \pi_*, g \circ \pi_*\} = 0,$$

$$\{\lambda_X, f \circ \pi_*\} = (a(X)f) \circ \pi_*,$$

$$\{\lambda_X, \lambda_Y\} = \lambda_{[X,Y]}.$$

Central fact used implicitly:

$$\mathfrak{g} \cong \mathfrak{g}^{**}$$

Another fact used implicitly: there exist functions with compact support (localizability)

Another fact used implicitly: Leibniz rule implies the existence of Poisson tensor

No longer true in general for infinite dimensional Lie algebras (e.g. Banach Lie algebras)

Definition

Banach Poisson manifold is a smooth manifold M modelled on Banach space with (localizable) Poisson bracket

$$\{f,g\} := \pi(df, dg)$$

given by a Poisson tensor $\pi \in \Gamma^\infty \bigwedge^2 T^{**}M$, such that the map $\sharp : T^*M \to T^{**}M$

$$\sharp\mu := \pi(\cdot,\mu)(m) \in T^{**}_m M, \quad \mu \in T^*_m M$$

takes values in tangent bundle

$$\sharp(T^*M) \subset TM.$$

Note 1: assumption of localizability means that one can apply Poisson bracket also to locally defined functions. It isn't the case in general even for Banach spaces due to the lack of smooth bump functions.

Note 2: there are Poisson brackets which are not defined by Poisson tensor and they might include higher order derivatives. In that case the usual approach to mechanics fails. Leibniz property does not imply the existence of Poisson tensor or the map $\sharp : T^*M \to T^{**}M$. Uniqueness is also not guaranteed.

Those pathological (queer) Poisson bracket exist e.g. on l^p spaces for $1 \leqslant p \leqslant 2$.

Note 3: condition $\sharp(T^*M) \subset TM$ ensures that hamiltonian vector fields are really vector fields and not sections of $T^{**}M$.

Definition

Banach space $\mathfrak{b}$ is a Banach Lie–Poisson space if $\mathfrak{b}^*$ is Banach Lie algebra with the propriety

$$\operatorname{ad}^*_x \mathfrak{b} \subset \mathfrak{b} \subset \mathfrak{b}^{**} \qquad \text{for } x \in \mathfrak{b}^*.$$

The Poisson bracket on $\mathfrak{b}$ is

$$\{f, g\}(b) := \langle [f'(b), g'(b)], b \rangle,$$

where $f'(b), g'(b) \in \mathfrak{b}^*$ are derivatives at point b.

Hamilton equations on $\mathfrak{b}$ with Hamiltonian $h \in C^\infty(\mathfrak{b})$

$$\frac{d}{dt} b = -\operatorname{ad}^*_{Dh(b)} b$$

Model example: trace-class operators $L^1(\mathcal{H})$

- It is predual to bounded operators:

$$(L^1(\mathcal{H}))^* = L^\infty(\mathcal{H})$$

- Since L^1 is an ideal we get:

$$\operatorname{ad}^*_X \rho = [X, \rho] \in L^1(\mathcal{H})$$

  for $X \in L^\infty(\mathcal{H})$ and $\rho \in L^1(\mathcal{H})$

- $\frac{d}{dt}\rho = -[h'(\rho), \rho]$
- For $\rho = |\psi\rangle\langle\psi|$ we get

$$\frac{d}{dt}\,|\psi\rangle = h'(\psi)\,|\psi\rangle$$

Definition

Let E be a Banach vector bundle over M. The structure of a Banach Lie algebroid on E is given as a localizable Lie bracket of sections $[\cdot,\cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ and a smooth bundle morphism covering identity (called *anchor*) $a : E \to TM$ such that

1. $[X, fY] = a(X)f \cdot Y + f[X, Y]$ for all $X, Y \in \Gamma(E), f \in C^\infty(M)$,
2. $(\Gamma(E), [\cdot,\cdot])$ is a Banach Lie algebra.
3. $[\cdot,\cdot]$ depends only on the first jet of sections.

A structure satisfying conditions 1. and 2. only would be called a *queer Banach Lie algebroid*. One could also expect the possibility of existence of such Banach Lie algebroids, but it turns out that for a wide class of Banach bundles no such situation can occur (it requires the existence of a Schauder basis).

Local picture

Local trivialization of E: open set $U \subset M$ such that $E_U := \pi^{-1}(U)$ is trivial, i.e.

$$E_U \cong U \times \mathbb{E}.$$

Proposition

Two sections $X, Y \in \Gamma(E_U)$ in trivialization $U \times \mathbb{E}$:

$$X(m) = (m, v_m) \qquad Y(m) = (m, w_m).$$

The local expression of Lie bracket of $E \to M$ is:

$$[X, Y](m) = \big(a(v_m)Y\big)(m) - \big(a(w_m)X\big)(m) + C_m(v_m, w_m)$$

for $m \in U \subset M$, where $U \ni m \mapsto C_m \in L_{skew}(\mathbb{E}, \mathbb{E}; \mathbb{E})$ is a certain smooth field of bounded skew-symmetric bilinear maps.

Attempt to construct linear Poisson structure on dual of E are not straightforward.

Problem

The differentials of functions $f \circ \pi^*$ and λ_X at some point ρ do not span the whole dual fiber $T^*_\rho E^*$.

Solution (Cabau–Pelletier)

Consider a (sub/weak/generalized) Poisson bracket defined only on a subspace of $\mathcal{A} \subset C^\infty(E^*)$.

Solution (G.-Jakimowicz)

Replace E^* with E_* (plus some conditions)

Theorem

The differentials of functions $f \circ \pi_*$ and λ_X, for $f \in C^\infty_{loc}(M)$, $X \in \Gamma_{loc}(E)$ span the cotangent bundle T^*E_* of the predual bundle E_*, i.e. for any $\rho \in E_*$ we have:

$$T^*_\rho E_* = \mathrm{span}\left(\{d(f \circ \pi_*)(\rho) \mid f \in C^\infty_{loc}(M)\} \cup \{d\lambda_X(\rho) \mid X \in \Gamma(E)\}\right).$$

Theorem

There exists a canonical linear Poisson bracket (localizable, non-queer) on E_* related to the Banach Lie algebroid E satisfying

$$\{f \circ \pi_*, g \circ \pi_*\} = 0,$$

$$\{\lambda_X, f \circ \pi_*\} = (a(X)f) \circ \pi_*,$$

$$\{\lambda_X, \lambda_Y\} = \lambda_{[X,Y]}.$$

Lemma

The sharp map $\sharp : T^*E_* \to T^{**}E_*$ in trivialization:

$$\sharp(m, \varphi, \mu, x) = \big(m, \varphi, -a(x), a^*(\mu) - (\mathrm{ad}^m_x)^*\varphi\big)$$

for $(m, \varphi, \mu, x) \in U \times \mathbb{E}_* \times \mathbb{M}^* \times \mathbb{E}$, where $a^* : T^*M \to E^*$ to the anchor and by $(\mathrm{ad}^m_x)^*$ we denote the dual map $(\mathrm{ad}^m_x)^* : \mathbb{E}^* \to \mathbb{E}^*$ to the map $\mathrm{ad}^m_x : \mathbb{E} \to \mathbb{E}$ given by $\mathrm{ad}^m_x(y) := C_m(x, y)$.

Theorem

The predual bundle E_* to the Lie algebroid E equipped with the canonical Poisson structure is a Banach Poisson manifold if and only if the following conditions hold

$$a^*(T^*M) \subset E_*$$

$$(\mathrm{ad}_x^m)^*(\mathbb{E}_*) \subset \mathbb{E}_*$$

Theorem

E_* — a Banach Poisson manifold with a linear Poisson bracket. There exists a canonically associated Banach Lie algebroid structure on E:

$$[X_1, X_2] = \lambda^{-1}\big(\{\lambda_{X_1}, \lambda_{X_2}\}\big)$$

$$a(X) = T\pi_* \sharp(d\lambda_X)$$

The mapping $\lambda : \Gamma_{loc}(E) \ni X \mapsto \lambda_X \in \mathcal{L}_{loc}(E_*)$ is a bijection.

Example: trivial bundle $E = \ell^2 \times \ell^\infty \to \ell^2$

Sections of E can be identified with maps $X : \ell^2 \to \ell^\infty$.
We fix a continuous linear map $A : \ell^\infty \to \ell^2$.
We define the bracket on such sections as:

$$[X, Y](x) = Y'(x)AX(x) - X'(x)AY(x).$$

We treat the derivative $X'(x)$ as a linear map $\ell^2 \to \ell^\infty$. The anchor in that case is

$$a(x, X) = (x, AX) \in \ell^2 \times \ell^2 \cong T\ell^2.$$

Proposition

The bundle $\ell^2 \times \ell^\infty$ with given anchor and Lie bracket is a Banach Lie algebroid.

Proposition

The predual bundle $\ell^2 \times \ell^1$ equipped with the canonical Poisson structure is a Banach Poisson manifold if and only if $A^*(\ell^2) \subset \ell^1$.

For example:

$$(AX)^k = \frac{1}{k} X^k$$