The algebraic approach to barycentric coordinates

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Barycentric coordinates

Let $\Omega \subseteq \mathbb{R}^d$ be an arbitrary polytope with vertices $v_1, \ldots, v_n$.

Barycentric coordinates with respect to Ω are the functions $b_i \colon \Omega \to \mathbb{R}$, $i = 1, \ldots, n$ s.t.

1. $\sum_{i=1}^{n} b_i(v) = 1$ (partition of unity);
2. $\sum_{i=1}^{n} b_i(v) v_i = v$ (linear precision);
3. $b_i(v_j) = \delta_{ij}$ (Lagrange property).

BC can be used as basis functions for **barycentric interpolation**: the function

$$f(v) = \sum_{i=1}^{n} b_i(v) f_i$$

interpolates the data f_i at the vertices v_i.

Non-negativity

The property

$$b_i(v) \geq 0$$

is required or just beneficial in many applications:

- **convex combinations** instead of **affine** ones;
- $f(v)$ lies inside the convex hull of the data f_i.

Barycentric coordinates cont.

- Introduced by A.F. Möbius, *Der Baryzentrische Calcul*, Barth, Leipzig, 1827.
- Unique for simplices;
- Convenient way to linearly interpolate data;
- Can be generalized in several ways to arbitrary polygons, polyhedra, higher dimensional polytopes, curves;
- Applications: **numerical analysis** (geometric modelling and computer graphics).

1. J. Warren, S. Schaefer, A.N. Hirani, M. Desbrun, *Barycentric cordinates for convex sets*, Adv. Comput. Math. **27** (2007), 319–338.
2. M.S. Floater, *Generalized barycentric coordinates and applications*, Acta Numer. **24** (2015), 161–214.

(Abstract) Algebra

$$(A, F)$$

$A \neq \varnothing$, F - set of operations $f\colon A^k \to A$, $k \in \mathbb{N}$.

- groups, rings, fields;
- vector spaces

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Here we are interested in:

- affine spaces
- convex sets
- barycentric algebras

Affine spaces of $\mathbb{R}^n$

The line $L_{x,y}$ through $x, y \in \mathbb{R}^n$:

$$L_{x,y} = \{\underline{p}(x,y) := (1-p)x + py \in \mathbb{R}^n \mid p \in \mathbb{R}\}.$$

A subset $A \subseteq \mathbb{R}^n$ is a (non-trivial) **affine subspace of $\mathbb{R}^n$** if together with any two different points x and y it contains the line $L_{x,y}$.

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Elements of the affine hull of a set $\{x_1, \ldots, x_n\}$ of affinely independent elements as **affine combinations**:

$$x = \sum_{i=1}^{n} r_i x_i \text{ with } \sum_{i=1}^{n} r_i = 1,$$

where $r_i \in \mathbb{R}$. They may be obtained by composing basic binary operations $\underline{p}$ and they form an affine space isomorphic to $\mathbb{R}^n$.

Convex sets of $\mathbb{R}^n$

Let I° be the open interval $]0,1[\subset \mathbb{R}$.

- The line segment $I_{x,y}$ joining the points x, y of $\mathbb{R}^n$:

$$I_{x,y} = \{\underline{p}(x,y) \mid p \in I^\circ\}.$$

- A subset $C \subseteq \mathbb{R}^n$ is a (non-trivial) **convex subset of $\mathbb{R}^n$** if together with any two different points x and y it contains the line segment $I_{x,y}$.
- **Convex sets** as **algebras**: $(C, \underline{I}^\circ)$, where $\underline{I}^\circ = \{\underline{p} \mid p \in I^\circ\}$.
- Convex polytopes = finitely generated convex sets.

Barycentric algebras

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Let R be a subfield of $\mathbb{R}$, $I = [0,1]$ closed unit interval of R,
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1. **complementation**: $r' = 1 - r$;
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Barycentric algebra over R is an algebra $(A, \underline{I}^\circ)$ equipped with binary operations $\underline{p}\colon A \times A \to A,\ (a,b) \mapsto \underline{p}(a,b)$ satisfying the identities:

1. **idempotence**: $\underline{p}(a,a) = a$;
2. **skew commutativity**: $\underline{p}(a,b) = \underline{p'}(b,a)$;
3. **skew associativity**: $\underline{p}(\underline{r}(a,b),c) = \underline{p \circ r}(a, \underline{\tfrac{p}{p \circ r}}(b,c))$.

Barycentric algebras and convex sets

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Convex sets are **cancellative** barycentric algebras, i.e. they satisfy the quasi-identities:

$$\underline{p}(a, b) = \underline{p}(a, c) \Rightarrow b = c$$

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W.D. Neumann, *On the quasivariety of convex subsets of affine spaces*, Arch. Math. **21** (1970) 11–16:

Some history

convex sets, convex modules, convexors, semiconvex sets, convex spaces...

- K. Keimel, G.D. Plotkin, *Mixed powerdomains for probability and nondeterminism*, Log. Methods Comput. Sci. **13** (2017) - **Remark 2.9 Historical Notes and References**:

- A.B. Romanowska, J.D.H. Smith, *Modal Theory* (1985) and *Modes* (2002).

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  M.H. Stone (1949) and H. Kneser (1952) - axiomatization of convex sets embaddable into vector spaces over linearly ordered skew fields (for barycentric algebras to have such property one has to add a cancellation axiom). Abstract convex sets = barycentric algebras.

- A.B. Romanowska, J.D.H. Smith, *Modal Theory* (1985) and *Modes* (2002).

Examples

1. Let $R = \mathbb{R}$ and V - a vector space over $\mathbb{R}$.
   Let $\underline{p}\colon V \times V \to V$ for $p \in I^{\circ}$ be the weighted mean operation:

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   We obtain the algebra $(V, \underline{I}^\circ)$ and its subalgebras are the convex sets of the real vector space V.
2. Let $(A, \vee)$ be a (join) semilattice: $a \vee b = b \Leftrightarrow a \leq b$.
   It becomes a barycentric algebra $(A, \underline{I}^\circ)$ if one defines:
   $$\underline{p}(a, b) = a \vee b$$
   for all $\underline{p} \in \underline{I}^\circ$ - **iterated semilattice**.

Applications

- Barycentric algebras unify ideas of **convexity** and **order**.
- Natural applications can be found in:
  - the modeling of systems that **function on** (potentially incomparable) **multiple levels**.

    J.D.H. Smith, *On the Mathematical Modeling of Complex Systems*, Center for Advanced Studies, Warsaw University of Technology, Warsaw, 2013.
  - theoretical computer science: both nondeterministic and probabilistic systems for verification.

    F. Bonchi, A. Sokolova, V. Vignudelli, *The Theory of Traces for Systems with Nondeterminism, Probability, and Termination*, Log. Methods Comput. Sci. **18** (2022).
  - thermostatic systems.

    J.C. Baez, O. Lynch, J. Moeller, *Compositional thermostatics*, J. Math. Phys. **64**, 023304 (2023).
  - in computational geometry to analyze systems of **barycentric coordinates**.

Simplices

***n*-dimensional simplex** Δ_n = a polytope with $n+1$ affinely independent vertices: $\mathbf{v}_0, \dots, \mathbf{v}_n$.

Each $\mathbf{x} \in \Delta_n$ may be expressed uniquely as a **convex combination** of vertices

$$\mathbf{x} = r_0\mathbf{v}_0 + \cdots + r_n\mathbf{v}_n \text{ with } r_i \in I \text{ and } \sum_{i=0}^{n} r_i = 1,$$

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- If $\mathbf{x}$ and $\mathbf{v}_i$ are given by Cartesian coordinates of $\mathbb{R}^n$, the barycentric coordinates r_i may be calculated by solving the above equation.
- Every polytope P with $n+1$ vertices is a homomorphic image of the simplex Δ_n. Hence each of its elements can also be presented by the above convex combination, however not in a unique way.

Simplices and free algebras

W. Neumann (1970): n-dimensional simplex is the **free barycentric algebra over a set of $n+1$ free generators**, i.e.

each function $f\colon V \to C$ from the generating set V of Δ_n to (the underlying set of) a barycentric algebra C has a unique extension to a barycentric homomorphism $\overline{f}\colon \Delta_n \to C$.

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Any n-dimensional polytope P with $k > n+1$ vertices is a homomorphic image of the simplex Δ_k.

Main goals

- bring an algebraic perspective to barycentric coordinates, based on barycentric algebras;
- introduce a general framework for coordinate systems on polytopes;
- compare different coordinate systems on convex polygons;
- introduce new coordinate systems on convex polygons.

Function spaces

- $(B, \underline{I}^\circ) \in \mathcal{B}$, X - set. B^X - **the space of all functions** $X \to B$, inherits barycentric algebra structure carried by B:
  for $a \in X$, $p \in I^\circ$, $f, g \colon X \to B$

$$(\underline{p}(f, g))(a) = \underline{p}(f(a), g(a)).$$

Polygon coordinate systems

Let Π be a polygon: $(\Pi, \underline{I}^\circ) \leq (\mathbb{R}^2, \underline{I}^\circ)$, with ordered vertex set $V = \{v_1, \dots, v_n\}$. A **coordinate system** for Π is a map

$$\lambda \colon V \to I^\Pi; v \mapsto \lambda_v$$

such that $a = \sum_{v \in V} \lambda_v(a) v_i$ (linear precision property) holds, for all $a \in \Pi$.

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- Partition of unity property: $\sum_{v \in V} \lambda_v(a) = 1$ follows from linear precision property;
- **BA:** $((I^\Pi)^V, \underline{I}^\circ) \simeq (I^{\Pi \times V}, \underline{I}^\circ) \in \mathcal{B}$.

Polygon coordinate systems

Theorem

The set K_Π of cooordinate systems on a polygon Π with vertex set V forms a **convex subset** *of $I^{\Pi \times V}$ under pointwise barycentric operations.*

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Tools for

- comparing different coordinate systems (discrepancy fields);
- introducing new coordinate systems (cartographic coordinates)

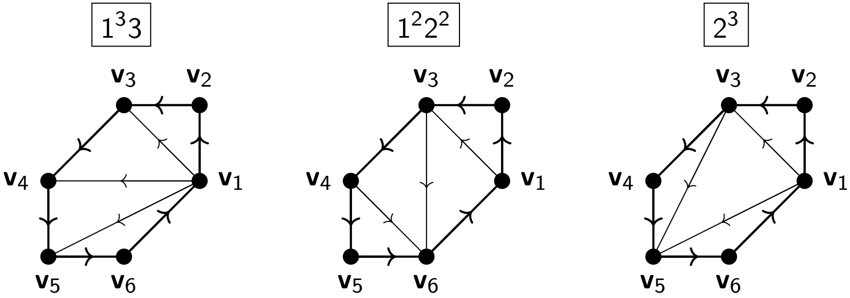
Sparse BC

- Local approach: **cartographic coordinates**.
- **The key idea:**
  1. decompose the polytope into simplices (*regions*)
  2. take the volumetric coordinates for the region within which a given point of the polytope lies.
- Any bias introduced by a particular decomposition may be removed by taking the average of a point's coordinates in each of the decompositions appearing in the orbit of a symmetry group.

Mathematical description

- Cyclic graph C_n constituted by the vertices and undirected edges of the polygon: **skeleton** of the polygon Π.
- **Chord** in C_n: edge connecting vertices which are not adjacent.
- **Chordal decomposition:** system of $n-3$ non-crossing chords of C_n that decompose Π as a union of $n-2$ simplices (triangles) whose vertices are vertices of Π.
- The $n-2$ triangles constitute the regions of the decomposition.
- Given any one chordal decomposition, we obtain others by the action of the dihedral group D_n as the automorphism group of the graph C_n.
- Leaving fixed the vertex set V of the polygon Π, the elements of D_n act on the $n-3$ chords of the decomposition.

Oriented, non-crossing chordal decompositions of hexagons



- 3 distinct decompositions of a hexagon into 4 triangles by means of three non-crossing chords;
- full set of representatives for the orbits of the dihedral group D_6 on the chordally subdivided graph C_6.

Cartographic coordinates

Definition

Let δ be a chordal decomposition of Π with a specified CDS. Then the formula

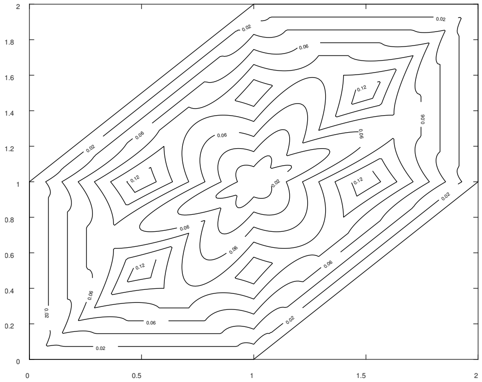
$$\kappa_v = \frac{1}{2n} \sum_{g \in D_n} g\delta_v$$

gives the **cartographic coordinate function** of that CDS at a vertex v of Π.

Theorem

For each chordal decomposition δ of Π, Definition above specifies a coordinate system κ of Π.

C−C discrepancy fields



Contour plot of the norm of the CDS $1^2 2^2 - 2^3$ discrepancy vector for a hexagon.

Further research

- cartographic coordinates as bounds on polygonal **BC**.

  M.S. Floater, K. Hormann, G. Kós, *A general construction of barycentric coordinates over convex polygons*, Adv. Comput. Math. **24** (2006), 311–331.

- use of **BC** to compute the electrostatic potential that is created by a charged triangular plate and analysis of the case of a charged polygonal plate by means of the principle of superposition, based on a single triangular decomposition of the polygon.

  U.-R. Kim, W. Han, D.-W. Jung, J. Lee, Ch. Yu, *Electrostatic potential of a uniformly charged triangle in barycentric coordinates*, Eur. J. Phys. **42** (2021), 045205 (24pp.).

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THANK YOU!