Subtleties of infinite-dimensional Poisson Geometry

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Motivation

Korteweg-de Vries Equation :

$$\dot{u} = u_{xxx} - 6uu_x$$

- bihamiltonian structure, hence by Magri-Lenard scheme [Magri '78], infinitely many conserved quantities
- Lax formulation [Kosmann-Schwarzbach, Magri '96]
- Equation of geodesics on the Virasoro group for a right invariant H^1-metric [Khesin-Misiolek '02]
- related to Poisson-Lie group of pseudodifferential operators [Khesin, Zakharevich '95]
- related to the restricted Grassmannian [Segal-Wilson'85]

Korteweg-de Vries in Lax form

$$\dot{u} = u_{xxx} - 6uu_x$$

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} - 6u\frac{\partial u}{\partial x}$$

$$\Leftrightarrow \frac{\partial L}{\partial t} = [P, L]$$

where

- $L = D^2 - 4u$
- $P = D^3 - \frac{3}{16}(uD + Du)$
- $D = \frac{\partial}{\partial x}$

In fact

$$P = (L^{\frac{3}{2}})_+$$

The restricted Grassmannian

$H = L^2(\mathbb{S}^1, \mathbb{C})$
$H = H_+ \oplus H_-$
$H_+ = \{f \in H, f(z) = a_0 + a_1 z + a_2 z^2 + \dots\}$ where $z = e^{i\theta}$
$H_- = \{f \in H, f(z) = a_{-1} z^{-1} + a_{-2} z^{-2} + a_{-3} z^{-3} + \dots\}$

$B \in GL(H_\pm, H_\pm)$ is Hilbert-Schmidt iff $\mathrm{Tr} B^* B < +\infty$

The restricted Grassmannian Gr_{res} **:** A closed subspace W of H belongs to the restricted Grassmannian Gr_{res} iff

1. $p_- : W \to H_-$ is Hilbert-Schmidt,
2. $p_+ : W \to H_+$ is Fredholm

The restricted Grassmannian

$$GL_{res} = \{\left\{ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right\} \in GL(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$P_{res} = \{\left\{ \begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix} \right\} \in GL(H), B \text{ is } \text{ Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = GL_{res}/P_{res}$$

$$U_{res} = \{\left\{ \begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right\} \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$$

$$\Rightarrow Gr_{res} = U_{res}/(U(H_+) \times U(H_-))$$

Triangular group B^+_{res} :

An invertible operator $g \in GL_{res}$ belongs to B^+_{res} if it is upper triangular with respect to the basis $\{z^{-n}, \dots, z^{-1}, 1, z, z^2, \dots\}$ of H, with strictly positive coefficents on the diagonal.

Remark : B^+_{res} acts on Gr_{res}

Relation between the restricted Grassmannian and the KdV hierachy

$\Gamma_+ = \{g = e^f, f \text{ holomorphic in } \mathbb{D}, f(0) = 0\}$
$\Rightarrow g = e^{t_1 z + t_2 z^2 + t_3 z^3 + \cdots} \in \Gamma_+$ acts on $L^2(\mathbb{S}^1, \mathbb{C})$ by multiplication and the corresponding operator is a Toeplitz upper triangular operator in B^+_{res}.

Proposition [SW85]

The action of Γ_+ on $\mathrm{Gr}^{(n)} = \{W \in \mathrm{Gr}^0_{\mathrm{res}}(\mathcal{H}) \ : z^n W \subset W\} \subset \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ induces the r-th KdV flow of the n-th hierarchy.

Key Observation : $\Gamma_+ \subset \mathrm{B}^+_{\mathbf{res}}(\mathcal{H})$.

Theorem [T]: The restricted Grassmannian

$$\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{U}_{\mathrm{res}}(\mathcal{H})/\,\mathrm{U}(\mathcal{H}_+) \times \mathrm{U}(\mathcal{H}_-) = \mathrm{GL}_{\mathrm{res}}(\mathcal{H})/\,\mathrm{P}_{\mathrm{res}}(\mathcal{H})$$

carries a natural Poisson structure such that :

1. the canonical projection $p\ :\mathrm{U}_{\mathrm{res}}(\mathcal{H}) \to \mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ is a Poisson map,
2. the natural action of $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ by left translations is a Poisson map,
3. the following right action of $\mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$ on $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) = \mathrm{GL}_{\mathrm{res}}(\mathcal{H})/\,\mathrm{P}_{\mathrm{res}}(\mathcal{H})$ is a Poisson map :

$$\begin{array}{ccc} \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \times \mathrm{B}^+_{\mathrm{res}}(\mathcal{H}) & \to & \mathrm{Gr}_{\mathrm{res}}(\mathcal{H}) \\ (g\,\mathrm{P}_{\mathrm{res}}(\mathcal{H}), b) & \mapsto & (b^{-1}g)\,\mathrm{P}_{\mathrm{res}}(\mathcal{H}). \end{array}$$

4. $\Gamma^+ \subset \mathrm{B}^+_{\mathrm{res}}(\mathcal{H})$ acts in a Poisson fashion on $\mathrm{Gr}_{\mathrm{res}}(\mathcal{H})$ and generated the KdV Hierarchy

A.B.Tumpach, *Banach Poisson-Lie groups and Bruhat-Poisson structure of the restricted Grassmannian*, Communications in Mathematical Physics, 2020.

Finite-dimensional Poisson manifolds

M a finite-dimensional manifold.

Poisson bracket = bilinear $\{\cdot,\cdot\} \; : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ with

- skew-symmetry $\{f,g\} = -\{g,f\}$
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$

Associated equations of motion: $\frac{df}{dt} = \{H,f\}$

Example: Hamilton's equations for $H = \frac{p^2}{2m} + V(q)$

Poisson tensor $P = \frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}$

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \text{ and } \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q}$$

Definition of a Poisson-Lie group

A **Poisson-Lie group** is a Lie group equipped with a Poisson manifold structure compatible with the group multiplication.

Finite-dimensional Poisson–Lie groups

Let us start with an example of a Manin triple...

$\mathfrak{u}(n)$ = Lie-algebra of the unitary group $U(n)$
= space of skew-symmetric matrices

$\mathfrak{b}(n)$ = Lie-algebra of the Borel group $B(n, \mathbb{C})$
= space of upper triangular matrices with real coef. on diagonal

Then the space $M(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$ of all complex matrices decomposes :

$$M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$$

and for the non-degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \operatorname{Im} \operatorname{Tr}(AB) = \text{ imaginary part of trace}(AB)$$

the blocks $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ are both isotropic.

Example

- $M(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$ with $\langle A, B\rangle = \mathrm{Im}\,\mathrm{Tr} AB$ is a Manin triple.
- U(n) and B($n, \mathbb{C}$) are dual Poisson-Lie groups with

$$\pi(R_g^* x_1, R_g^* x_2) = \langle p_{\mathfrak{u}}(g^{-1} x_1 g), p_{\mathfrak{b}}(g^{-1} x_2 g)\rangle$$

- Moreover $GL(n,\mathbb{C}) = U(n) \times B(n)$ because of Iwasawa dec.
- This gives a dressing action $\varphi : B(n) \times U(n) \to U(n)$ by $\varphi(b)(k) = k'$ where k' is the unique element of $U(n)$ such that $bk = k'b'$ with $b' \in B(n)$.
- the Grassmannians Gr(p, n) = U(n)/($U(p) \times U(n-p)$) are Poisson homogeneous spaces
- the right action of B($n, \mathbb{C}$) on Gr(p, n) is a Poisson map
- the symplectic leaves of Gr(p, n) are the orbits under the action of B($n, \mathbb{C}$)

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Poisson manifold modeled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathcal{C}^1$ away from the origin)

(2) there exists derivations of order greater then 1, called Queer operational tangent vectors [Kriegl, Michor, '97]

(3) there exists Poisson bracket without Poisson tensor, called Queer Poisson bracket (Leibniz rule does not imply existence of Poisson tensor) [Beltita, Golinski, T., 2018]

(4) existence of Hamiltonian vector field is not automatic

Definition

Consider a unital subalgebra $\mathcal{A} \subset \mathcal{C}^\infty(M)$ of smooth functions on a Banach manifold M, i.e. $\mathcal{A}$ is a vector subspace of $\mathcal{C}^\infty(M)$ containing the constants and stable under pointwise multiplication. An $\mathbb{R}$-bilinear operation $\{\cdot,\cdot\} : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is called a **Poisson bracket** on the set of **admissible functions** $\mathcal{A}$ if it satisfies :

(i) anti-symmetry : $\{f, g\} = -\{g, f\}$;

(ii) Jacobi identity : $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$;

(iii) Leibniz formula : $\{f, gh\} = \{f, g\}h + g\{f, h\}$;

Let us recall the definition of a duality pairing.

Definition

Let $\mathfrak{g}_+$ and $\mathfrak{g}_-$ be two normed vector spaces over the same field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, and let

$$\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$$

be a continuous bilinear map. One says that the map $\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-}$ is a **duality pairing** between $\mathfrak{g}_+$ and $\mathfrak{g}_-$ if and only if it is **non-degenerate**, i.e. if the following two conditions hold :

$$\left(\langle x, y \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} = 0, \quad \forall x \in \mathfrak{g}_+\right) \Rightarrow y = 0 \quad \text{and}$$
$$\left(\langle x, y \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} = 0, \quad \forall y \in \mathfrak{g}_-\right) \Rightarrow x = 0.$$

Definition

Let M be a Banach manifold and $\mathcal{A}$ be a unital subalgebra of $\mathcal{C}^\infty(M)$. The first jet of $\mathcal{A}$, denoted by $\mathrm{J}^1(\mathcal{A})$ is the subbundle of the cotangent bundle T^*M whose fiber over $p \in M$ is the space of differentials of functions in $\mathcal{A}$,

$$\mathrm{J}^1(\mathcal{A})_p = \{df_p \ : f \in \mathcal{A}\}.$$

We will say that the unital subalgebra $\mathcal{A}$ is **large** if the duality pairing between TM and T^*M restricts to a duality pairing between TM and $\mathrm{J}^1(\mathcal{A})$.

A unital subalgebra $\mathcal{A}$ of $\mathcal{C}^\infty(M)$ is large if and only if, for every $p \in M$, the pairing

$$\begin{array}{rcccl} \langle \cdot, \cdot \rangle_{J^1(\mathcal{A}), TM} & : & J^1(\mathcal{A})_p \times T_pM & \to & \mathbb{R} \\ & & (df_p, X) & \mapsto & \langle df_p, X \rangle_{TM^*, TM} \end{array}$$

is non-degenerate. This is the case if and only if $\mathrm{J}^1(\mathcal{A})_p$ separates points in the tangent space T_pM for any $p \in M$, i.e. if for every $p \in M$, a tangent vector $X \in T_pM$ such that

$$\langle df_p, X \rangle_{T^*M, TM} = 0, \quad \forall f \in \mathcal{A}$$

necessarly vanishes.

Definition

Let M be a Banach manifold endowed with a Poisson bracket on the space of admissible functions $\mathcal{A}$. A function $h \in \mathcal{A}$ is called **Hamiltonian** if the derivation $\{\cdot, h\}$ is represented by a smooth vector field, i.e. if there exists a smooth vector field X_h on M such that

$$X_h(f) := \langle df, X_h \rangle_{TM^*,TM} = \{f, h\},$$

for any $f \in \mathcal{A}$, in particular $\{\cdot, h\}$ is a first order operator. One calls X_h a Hamiltonian vector field associated to f.

Proposition:

Let M be a Banach manifold endowed with a Poisson bracket on the space of admissible functions $\mathcal{A}$. If $\mathcal{A}$ is large, then any Hamiltonian function $h \in \mathcal{A}$ is associated to a **unique** Hamiltonian vector field X_h.

Proof.

The uniqueness of the Hamiltonian vector field is a consequence of the hypothesis that $\mathcal{A}$ is large. Indeed, let X_1 and X_2 be two Hamiltonian vector fields associated to $h \in \mathcal{A}$. Then

$$\begin{aligned}(X_1 - X_2)(f) &= \langle df, X_1 - X_2\rangle_{TM^*,TM}\\ &= \langle df, X_1\rangle_{TM^*,TM} - \langle df, X_2\rangle_{TM^*,TM}\\ &= \{f, h\} - \{f, h\} = 0,\end{aligned}$$

for any $f \in \mathcal{A}$. Since $\mathcal{A}$ is large, the differentials of functions in $\mathcal{A}$ separates points in TM. It follows that $X_1 - X_2 = 0$. □

Proposition

Let M be a Banach manifold endowed with a Poisson bracket on the space of admissible functions $\mathcal{A}$. The space of Hamiltonian functions is a unital subalgebra of $\mathcal{A}$.

Proof.

This is a direct consequence of the Leibniz rule. □

Definition

Let M be a Banach manifold endowed with a Poisson bracket π on the space of admissible functions $\mathcal{A}$. One says that $\mathcal{A}$ is **Hamiltonian** if any f in $\mathcal{A}$ is Hamiltonian.

A Banach manifold M endowed with a Poisson bracket π on a space of admissible functions $\mathcal{A}$ with $\mathcal{A}$ large and Hamiltonian is a weak Banach Poisson manifold in the sense of [Neeb-Sahlmann-Thiemann] . Given any Banach manifold M endowed with a Poisson bracket on a space of admissible functions $\mathcal{A}$, one can always restricts the Poisson bracket to the space of Hamiltonian functions, but the space of Hamiltonian functions may be empty, or not large.

Alternative definition of a Banach Poisson manifold

Definition of a Poisson tensor :
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\,(\pi(\alpha, \beta))$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\,(\alpha, d\,(\pi(\beta, \gamma))) + \pi\,(\beta, d\,(\pi(\gamma, \alpha))) + \pi\,(\gamma, d\,(\pi(\alpha, \beta))) = 0.$$

Definition of a Poisson Manifold : A Banach (sub-)Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Banach symplectic manifold

Any Banach symplectic manifold (M, ω) is naturally a Banach (sub-)Poisson manifold $(M, \mathbb{F}, \pi)$ with

1. $\mathbb{F} = \omega^\sharp(TM)$;
2. $\pi \ : \omega^\sharp(TM) \times \omega^\sharp(TM) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \omega(X_\alpha, X_\beta)$ where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Definition

Consider a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} \ : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$ between 2 Banach. $\mathfrak{g}_+$ is a **Banach Lie–Poisson space with respect to** $\mathfrak{g}_-$ if

- $\mathfrak{g}_-$ is a Banach Lie algebra $(\mathfrak{g}_-, [\cdot, \cdot]_{\mathfrak{g}_-})$
- $\mathfrak{g}_-$ acts continuously on $\mathfrak{g}_+ \hookrightarrow \mathfrak{g}_-^*$ by coadjoint action, i.e.

$$\mathrm{ad}^*_\alpha x \in \mathfrak{g}_+,$$

for all $x \in \mathfrak{g}_+$ and $\alpha \in \mathfrak{g}_-$, and $\mathrm{ad}^* \ : \mathfrak{g}_- \times \mathfrak{g}_+ \to \mathfrak{g}_+$ is continuous.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

$\Updownarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Uparrow$

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

Some challenges in infinite-dimensional Poisson Geometry...

1. **What are the possible local models for Poisson geometry? for symplectic geometry?**
2. **What about Weinstein's Splitting Theorem?** (Weinstein, Eyni,...)
3. **What is the link between hyperbolic geometry, integrable systems and Poisson Geometry?** (see Seppi, Takhtajan, Tenenblat, Teo, Wang...)
4. **Define Poisson integrators for PDE's using Banach algebroids and groupoids** (see Beltita, Cosserat, Dobrogowska, Golinski, Jakimowicz, Laurent-Gengoux, Pelletier, Odzijewicz, Salnikov, Slizewska...)

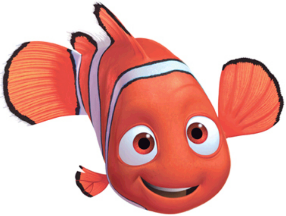
References for Symplectic and Poisson geometry on infinite-dimensional manifolds

Wurzbacher, 2001
A. A. Odzijewicz, T. Ratiu, 2003
Haller, Vizman, 2004, 2020, 2022
Gay-Balmaz, Vizman, 2010, 2014, 2016, 2020
P. Cabau, F. Pelletier, 2011, 2024
Balleier, Wurzbacher, 2012
K.H.Neeb, H. Sahlmann, T. Thiemann, 2013
Janssens, Neeb, 2015
de Bièvre, F.Genoud, S. Rota Nodari, 2015
Andruchow, Larotonda, 2008, 2018
D. Beltita, T. Golinski, A.B.Tumpach, 2018
Andruchow, Corach, Recht, 2019
A.B.Tumpach, 2020

- A.B.Tumpach, *Bruhat-Poisson structure of the restricted Grassmannian and the KdV hierarchy*, Communications of Mathematical Physics, 2020.
- D. Beltita, T. Golinski, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics.
- A.B.Tumpach, *Gauge invariance of degenerate Riemannian metrics*, Notices of AMS.
- A.B.Tumpach, H. Drira, M. Daoudi, A. Srivastava, *Gauge invariant Framework for shape analysis of surfaces*, IEEE TPAMI.
- D. Beltita, T. Ratiu, A.B. Tumpach, *The restricted Grassmannian, Banach Lie-Poisson spaces, and coadjoint orbits*, Journal of Functional Analysis.
- A.B.Tumpach, *Hyperkähler structures and infinite-dimensional Grassmannians*, Journal of Functional Analysis.
- A.B.Tumpach, *Infinite-dimensional hyperkähler manifolds associated with Hermitian-symmetric affine coadjoint orbits*, Annales de l'Institut Fourier.
- A.B.Tumpach, *Classification of infinite-dimensional Hermitian-symmetric affine coadjoint orbits*, Forum Mathematicum.

Poisson bracket not given by a Poisson tensor

Queer Poisson Bracket = Poisson bracket not given by a Poisson tensor



Reference :

D. Beltiță, T. Goliński, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics, 2018.

Poisson bracket not given by a Poisson tensor

$\mathcal{H}$ separable Hilbert space

Kinetic tangent vector $X \in T_x\mathcal{H}$ equivalence classes of curves $c(t)$, $c(0) = x$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Operational tangent vector $x \in \mathcal{H}$ is a linear map $D : C_x^\infty(\mathcal{H}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(fg)(x) = Df\ g(x) + f(x)\ Dg$$

Poisson bracket not given by a Poisson tensor

Ingredients :

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathcal{K}(\mathcal{H}) \subsetneq \mathcal{B}(\mathcal{H})$ bounded operators
  $\Rightarrow \exists \ell \in \mathcal{B}(\mathcal{H})^*$ such that $\ell(\mathrm{id}) = 1$ and $\ell_{| \mathcal{K}(\mathcal{H})} = 0$.

Queer tangent vector [Kriegl-Michor]

Define $D_x : C_x^\infty(\mathcal{H}) \to \mathbb{R}$, $D_x(f) = \ell(d^2(f)(x))$, where the bilinear map $d^2(f)(x)$ is identified with an operator $A \in \mathcal{B}(\mathcal{H})$ by Riesz Theorem

$$d^2(f)(x)(X, Y) = \langle X, AY \rangle$$

Then D_x is an operational tangent vector at $x \in \mathcal{H}$ of order 2

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

$$d(fg)(x) = df(x).g(x) + f(x).dg(x)$$

$$\begin{aligned} d^2(fg)(x) &= d^2f(x).g(x) + df(x) \otimes dg(x) \\ &\quad + dg(x) \otimes df(x) + f(x) d^2 g(x) \end{aligned}$$

$$\begin{aligned} D_x(fg) &= \ell(d^2(fg)(x)) \\ &= \ell(d^2 f(x)).g(x) + f(x)\ell(d^2 g(x)) \\ &\quad + \ell(df(x) \otimes dg(x)) + \ell(dg(x) \otimes df(x)) \\ &= D_x f \; g(x) + f(x) \; D_x g \end{aligned}$$

Poisson bracket not given by a Poisson tensor

Theorem (D. Beltita, T. Golinski, A.B.Tumpach)

Consider $\mathcal{M} = \mathcal{H} \times \mathbb{R}$. Denote points of $\mathcal{M}$ as (x, λ). Then $\{\cdot, \cdot\}$ defined by

$$\{f, g\}(x, \lambda) := D_x\left(f(\cdot, \lambda)\right) \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x\left(g(\cdot, \lambda)\right)$$

a queer Poisson bracket on $\mathcal{H} \times \mathbb{R}$, in particular it can not be represented by a bivector field $\Pi : T^*\mathcal{M} \times T^*\mathcal{M} \to \mathbb{R}$. The Hamiltonian vector field associated to $h(x, \lambda) = -\lambda$ is the queer operational vector field

$$X_h = \{h, \cdot\} = D_x$$

acting on $f \in C^\infty_x(\mathcal{H})$ by $D_x(f) = \ell(d^2(f)(x))$.