Conjugate points in the Grassmann manifold of a C^*-algebra

Who? GABRIEL LAROTONDA

From? Instituto Argentino de Matemática (CONICET) &
Universidad de Buenos Aires - Argentina.

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JOINT WORK WITH E. ANDRUCHOW AND L. RECHT

Definitions

$\mathcal{A}$	real or complex C^*-algebra
P	a self-adjoint projection $P = P^2 = P^* \in \mathcal{A}$
$\mathcal{U}_{\mathcal{A}}$	unitary operators of $\mathcal{A}$, $U^* = U^{-1}$
$\mathcal{A}_{sk}$	skew-hermitian operators $V^* = -V$
$\mathcal{A}_h$	hermitian operators $X^* = X$
$Gr(P_0)$	connected component of the Grassmann manifold of $\mathcal{A}$

That is, for fixed $P_0 = P_0^* = P_0^2 \in \mathcal{A}_h$

$$Gr = Gr(P_0) = \{UP_0U^* : U \in \mathcal{U}_{\mathcal{A}}\}.$$

Main points of the talk

1. natural connection ∇ (in different disguises) in Gr,
2. geodesics are described by $\gamma(t) = e^{tz} P e^{-tz}$ for $z^* = -z$ and $z = zP + Pz$
3. the exponential map of ∇ is then $\mathrm{Exp}_P(Z) = e^{[Z,P]} P e^{-[Z,P]}$
4. compatibility with the metric $\|\gamma'(t)\|_{\gamma(t)} = \|\gamma'(t)\|_\infty$
5. distance as infima of lengths of paths in Gr
6. *conjugate points* along geodesics in the Grassmannian:
   - in the metric sense (cut locus for the Finsler metric induced by the norm of $\mathcal{A}$)
   - in the tangent sense (the differential of the exponential map along γ is not invertible).

Part 1: linear connection ∇ in Gr

Fix $P \in Gr$, operators $A \in \mathcal{A}$ as 2×2 block matrices:

$$A = \begin{pmatrix} PAP & PAP^{\perp} \\ P^{\perp}AP & P^{\perp}AP^{\perp} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

and the algebra $\mathcal{A}$ decomposed as

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} = A_d + A_c,$$

$\mathcal{D}_P$ is the *P-diagonal part* of $\mathcal{A}$,

$\mathcal{C}_P$ is the *P-co-diagonal part* of $\mathcal{A}$.

1 $X \in \mathcal{D}_P$ iff it commutes with P

2 $X \in \mathcal{C}_P \iff X = XP + PX$, here $[P,[P,X]] = X$.

3 $X \in \mathcal{C}_P$ then $\sigma(X)$ is balanced ($\lambda \in \sigma$ iff $-\lambda \in \sigma$)

4 $X \in \mathcal{C}_P \iff UXU^* \in \mathcal{C}_{UPU^*}$, for any $U \in \mathcal{U}_{\mathcal{A}}$.

5 $Gr \subset \mathcal{A}_h$. Tangent space $T_P Gr = \mathcal{C}_P \cap \mathcal{A}_h$

Typical tangent vector at P: $X_P = [x, P]$ with $x^* = -x \in \mathcal{C}_P$
Such x is unique, $x = [X_P, P]$. Correspondence

6 $$\mathcal{C}_p \cap \mathcal{A}_{sk} \qquad \longleftrightarrow \qquad \mathcal{C}_P \cap \mathcal{A}_h$$

$$x = \begin{pmatrix} 0 & -\lambda \\ \lambda^* & 0 \end{pmatrix} \xleftrightarrow{-\operatorname{ad} P = [\,\cdot\,,P\,]} \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix} = X$$

Connection ∇

Project

onto the tangent spaces

$$\mathcal{A}_h \ni V^* = V = \begin{pmatrix} a & \lambda \\ \lambda^* & b \end{pmatrix} \mapsto \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix} = \Pi_P(V) \in T_P Gr$$

$\langle\ ,\ \rangle$

If $\mathcal{A}$ has a faithful trace, Π_P are the orthogonal projections for the Riemannian metric

$$\langle X, Y \rangle = Tr(XY) \qquad X, Y \in \mathcal{A}_h$$

$\mu : [0,1] \to Gr$ a *vector field along a path* $\gamma \subset Gr$ i.e. $\mu(t) \in T_{\gamma(t)} Gr = \mathcal{C}_{\gamma(t)}$ for each $t \in [0,1]$.

∇

$$D_t \mu := \Pi_{\gamma(t)}(\mu'(t)) \qquad \text{covariant derivative of } \mu$$

$\mathcal{A}$ with trace: D_t is the Levi-Civita connection.

Geodesics and exponential map

Fix $P \in Gr$, $Z = [z, P] \in T_P Gr$, then

$$\delta(t) = e^{tz} P e^{-tz}$$

is the unique geodesic of the connection ∇ i.e

$$D_t \delta' = 0 \qquad \text{(Euler's equation)}$$

with

$$\delta(0) = P, \qquad \delta'(0) = Z = [z, P]$$

Thus the exponential map of (Gr, ∇) is

$$\mathrm{Exp}_P(Z) = \delta(1) = e^{z} P e^{-z} = e^{[Z,P]} P e^{-[Z,P]}$$

Paralell transport and the metric

The paralell transport equation along γ

$$D_t\mu = 0,\ \mu(0) = W \in T_P Gr \qquad \Longrightarrow \quad P_0^t(\gamma)W = \mu(t)$$

is solved explicitly when γ is a geodesic $t \mapsto e^{tz}Pe^{-tz}$: it is

$$\mu(t) = e^{tz}We^{-tz}$$

L and *dist*

Length of paths $\gamma : [0,1] \to Gr$ is $L(\gamma) = \int_0^1 \|\gamma'(t)\| dt$
$\|X\|$ is the C^*-algebra norm of X.

$$dist(P,Q) = \inf\{L(\gamma) : \gamma(0) = P, \gamma(1) = Q\}.$$

Compatibility of the connection with the metric:

$$\|P_0^t(\gamma)W\| = \|W\| \qquad dist(UPU^*, UQU^*) = dist(P,Q).$$

Part 2: Tangent conjugate locus

Fix P, for each $V \in T_P Gr$, we have the differential

$$(D\,\mathrm{Exp}_P)_V : T_P Gr \to T_{\mathrm{Exp}_P(V)} Gr = T_{e^v P e^{-v}} Gr$$

We define the *tangent conjugate locus at P* as

$$TCL = \{V \in T_P Gr : D(\mathrm{Exp}_P)_V \text{ is not an isomorphism}\}$$

- The map $D(\mathrm{Exp}_P)_{V=0}$ is a linear isomorphism of $T_P Gr$
- The *first tangent conjugate point* is the smaller V in the tangent conjugate locus of P
- The *cut locus* is the set of point $Q \in Gr$ such that geodesics from P are not minimizing past Q.

$(D\,\mathrm{Exp}_P)_V : T_P Gr \to T_{\mathrm{Exp}_P(V)} Gr$

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- The map $D(\mathrm{Exp}_P)_{V=0}$ is a linear isomorphism of $T_P Gr$
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In the classical (Riemannian, finite dimensional) setting:

Thm $Q = \mathrm{Exp}_P(t_0 V)$ is in the cut locus iff $t_0 V$ is the first tangent conjugate point from P in the direction of V.

Two formulas for $(D\,\mathrm{Exp}_P)_V$

$$D(Exp_P)_V W = e^v[\mathrm{sinhc}(\mathrm{ad}\, v)w, P]e^{-v}$$

$$V = [v, P], \quad W = [w, P], \quad \mathrm{ad}\, v(z) = [v, z]$$

$$\mathrm{sinhc}(\lambda) = \frac{\sinh(\lambda)}{\lambda} = \frac{e^{\lambda} - e^{-\lambda}}{2\lambda}$$

Now $X \mapsto e^v X e^{-v}$ is invertible

so is $z \mapsto [z, P]$ so we need to understand

$w \mapsto \mathrm{sinhc}(\mathrm{ad}\, v)w$, a self-map of $\mathcal{A}_{sk}$

$$\mathrm{sinhc}(t\, \mathrm{ad}\, v) = \Pi_{k\geq 1}\left(1 + \frac{t\, \mathrm{ad}^2\, v}{k^2\pi^2}\right)$$

by means of the Weierstrass factorization theorem for entire functions of finite order.

Conjugate points

$T \in \mathbb{R}$, $V = [v, P] \in \mathcal{C}_P$, TV tangent conjugate iff

$$\mathrm{sinhc}(T \,\mathrm{ad}\, v) = \Pi_{k\geq 1}\left(1 + \frac{T\,\mathrm{ad}^2 v}{k^2\pi^2}\right) \quad \text{is not invertible.}$$

monoconjugate if it is not injective

epiconjugate if it is not surjective

Theorem *(Andruchow-L-Recht) Normalize* $\|V\| = 1$, *then* $Q = \delta(T) = \mathrm{Exp}_P(TV)$ *is in TLC only if*

$$T = T(k, s, s') = \frac{k\pi}{|s - s'|}$$

$$k \in \mathbb{Z}_{\neq 0} \quad s \neq s' \in \sigma(V) \subseteq [-1, 1]$$

First conjugate point

$$T = \frac{k\pi}{|s - s'|}$$

First one is at $T = \frac{\pi}{2}$, since $s = \pm 1$ both belong to $\sigma(V)$

$$Q = \delta(\pi/2) = e^{\frac{\pi}{2}V} P e^{-\frac{\pi}{2}V}$$

The polar decomposition of $V = u|V|$ is written by blocks as

$$V = \begin{pmatrix} 0 & \lambda \\ \lambda^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Omega \\ \Omega^* & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\lambda\lambda^*} & 0 \\ 0 & \sqrt{\lambda^*\lambda} \end{pmatrix} = u|V|$$

with a partial isometry $\Omega : \mathrm{Ran}(1 - P) \to \mathrm{Ran}\, P$.

Theorem *(Andruchow-L-Recht 2023) First tangent cut locus*

$$TCL = \{\Omega z - z\Omega^* : \quad z^* = -z, \quad |\lambda| z = z\}$$

Moreover, if Q is not monoconjugate, then it is epiconjugate.

More conjugate points

$$\textit{Theorem}: \quad T = \frac{k\pi}{|s - s'|} \quad \text{full description}$$

- $T = \frac{k\pi}{2}$ is always conjugate
- "simple" examples, $\mathcal{A} = \mathcal{B}(H)$ with $H = L^2[-1,1]$ where Q
  1. is monoconjugate but not epiconjugate
  2. is epiconjugate but not monoconjugate
  3. both monoconjugate and epiconjugate
- For other $T = T(k, s, s')$: if $\mathcal{A}$ is a von Neumann factor or a prime C^*-algebra, it is always conjugate.
- Nice description in projective spaces ($dim(\mathrm{Ran}(P)) = 1$)

Less conjugate points $\mathcal{A} = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$

Fix $0 < \alpha < 1$. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}$$

Then V is P-codiagonal, $\sigma(V) = \{-1, -\alpha, \alpha, 1\}$.

There are four family of candidates to conjugate points,

$$T_1 = \frac{k\pi}{2}, \quad T_2 = \frac{k\pi}{1+\alpha}, \quad T_3 = \frac{k\pi}{1-\alpha}, \quad T_4 = \frac{k\pi}{2\alpha}.$$

For the first family we know that $\gamma(T_1)$ is conjugate to P.

But *none of the other points* $\gamma(T_i)$ *are conjugate to* P

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Thank you!