Q-manifolds and sigma models

Noriaki Ikeda
Ritsumeikan University
Kyoto, Japan

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§1. Introduction

Applications of graded manifolds and Q-manifolds, to physical theories, especially, to BV-formalism.

This talk is related to Vladimir Salnikov's lecture in the next week.

Purpose

Quantum field theories are not completely formulated as mathematics yet.
$\rightarrow$ Toward mathematical formulations

Examples and applications of graded manifolds

- Fermions, supersymmetry

- Ghosts in gauge theories (BRST-BV-BFV formalism)
$\rightarrow$ Canonical, path integral quantizations

- Poisson brackets、Lie algebras $\rightarrow$ Analytic mechanics

- Homological algebras and topological invariants based on differential complexes (anomalies, topological effects)

- L_∞-algebras, A_∞-algebras, (string field theory, etc.)

- Deformation quantizations and formality

- T-, S-, U-dualities in string theories
- Current algebras
- Variational principle (variational bicomplex)

etc.

Basic examples

In supersymmetry, $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$

Ghosts and BRST-BV formalism in quantum field theories

A vector bundle and differential forms are not sufficient!

Non-graded formulation $\longleftrightarrow$ Graded formulation

Analytical mechanics and gauge theories

Lagrangian formalism $\longleftrightarrow$ Batalin-Vilkovisky (BV) formalism

Hamiltonian formalism $\longleftrightarrow$ Batalin-Fradkin-Vilkovisky (BFV) formalism

Quantum field theories

Dirac quantizations $\longleftrightarrow$ BRST-BV quantizations

Plan of Talk

Batalin-Vilkovisky formalism (Poisson sigma model)

Q-manifolds (differential graded manifolds) and QP-manifolds

Geometry induced from graded manifolds

Recent developments

§2. BRST formalism to BV formalism

Yang-Mills (nonabelian gauge) theory

$$S = -\frac{1}{4}\int \mathrm{tr}(F \wedge *F), \qquad F = \mathrm{d}A + A \wedge A.$$

Infinitesimal gauge transformations are

$$\delta_\epsilon A = \mathrm{d}\epsilon + [A, \epsilon],$$

where ϵ is a gauge parameter. Then,

$$\delta_\epsilon S = 0, \qquad [\delta_\epsilon, \delta_{\epsilon'}]A = \delta_{[\epsilon,\epsilon']}A. \qquad \text{(off-shell)}$$

Historical developments of quantizations of gauge theories

Gauge fixing, Gupla-Bleuler, Faddev-Popov (FP) ghosts, 't-Hooft-Veltman, Becchi-Rouet-Stora-Tyutin (BRST), Kugo-Ojima…

BRST transformations Change $\epsilon(\sigma)$ to a Grassman odd field (FP ghost) $c(\sigma)$,

$$sA = \mathrm{d}c + [A, c].$$

ghost number: $\mathrm{gh}\, A = 0$, $\mathrm{gh}\, c = 1$. s is of ghost number one. If $sc = -\frac{1}{2}[c, c]$, we obtain

$$sS = 0, \qquad s^2 = 0. \qquad \text{(off-shell)}$$

BRST quantization Introduce the antighosts, odd and even $\bar{c}$, b such that $s\bar{c} = ib$ and $sb = 0$, and consider the gauge fixing,

$$S_q = \int \Big(\mathrm{tr}(F \wedge *F) + b * \mathrm{d} * A + *\frac{\alpha}{2}\mathrm{tr}(bb) - i\bar{c} * \mathrm{d} * Dc \Big) .$$

$sS_q = 0$ and $s^2 = 0$ (off-shell).

$$Z = \int_{\mathcal{L}} \mathcal{D}A\mathcal{D}b\mathcal{D}c\mathcal{D}\bar{c}\; e^{\frac{i}{\hbar}S_q}$$

is BRST invariant. $\mathcal{L}$ is a Lagrangian submanifold of the space of fields. Physical states are defined by $s|\mathrm{phys}\rangle = 0$.

Problems of BRST formalism

General consistency conditions of a classical gauge theory are

$$\delta_\epsilon S = 0 \ \ \text{(off shell)}, \qquad [\delta_\epsilon, \delta_{\epsilon'}] = \delta_{[\epsilon,\epsilon']} + \text{(equations of motion)}.$$

In the BRST formalism,

$$sS = 0 \ \ \text{(off shell)}, \qquad s^2 = \text{(equations of motion)}.$$

However, gauge fixings change the EOMs because the action functional changes S to S_q. $s^2 = \text{(equations of motion)}$ does not hold. The physical condition $s|\text{phys}\rangle = 0$ is inconsistent!

Batalin-Vilkovisky (BV) formalism Batalin-Vilkovisky '83,'85

By introducing auxiliary fields, we modify the action functional S to S_{BV} satisfying

$$sS_{BV} = 0 \ \ \text{(off shell)}, \qquad s^2 = 0 \ \ \text{(off shell)},$$

without changing physics, and quantize it. Quantization with gauge fixing is consistent,

$$Z \ = \ \int_{\mathcal{L}} \mathcal{D}A\mathcal{D}b\mathcal{D}c\mathcal{D}\bar{c}\ e^{\frac{i}{\hbar}S_{BVq}}$$

We can consistently impose the physical condition, $s|\text{phys}\rangle = 0$.

§3. Poisson sigma model

NI '93, Schaller and Strobl '94

A *sigma model* is a mechanics on a mapping space.

It is a sigma model from Σ in two dimensions with local coordinate σ^μ to a target manifold M in d dimensions.

$X^i : \Sigma \to M$, $A_i = A_{\mu i}(\sigma) \mathrm{d}\sigma^\mu$: gauge field

$$
\begin{aligned}
S &= \int_\Sigma \left(A_i \wedge \mathrm{d}_\Sigma X^i + \frac{1}{2} \pi^{ij}(X) A_i \wedge A_j \right) \\
&= \int_\Sigma \mathrm{d}^2\sigma \left(\epsilon^{\mu\nu} A_{\mu i} \partial_\nu X^i + \frac{1}{2} \epsilon^{\mu\nu} \pi^{ij}(X) A_{\mu i} A_{\nu j} \right)
\end{aligned}
$$

where $\pi^{ij}(X) = -\pi^{ji}(X)$ is an antisymmetric tensor.

This model describes

- two dimensional dilaton gravity (deformed JT gravity)
- A-model, B-model (topological string theory)
- BF type theories are used in generalized symmetries.
- Tree (disc) open string amplitude gives the Kontsevich's formula of the deformation quantization on a Poisson manifold.

Kontsevich '97, Cattaneo, Felder '99

The action functional is gauge invariant under the following gauge transformations,

$$\delta_\epsilon X^i = -\pi^{ij}(X)\epsilon_j, \qquad \delta_\epsilon A_i = \mathrm{d}\epsilon_i + \tfrac{1}{2}\frac{\partial \pi^{jk}(X)}{\partial X^i} A_j \epsilon_k,$$

iff π is a Poisson structure,

$$\frac{\partial \pi^{ij}}{\partial X^m}\pi^{mk} + \frac{\partial \pi^{jk}}{\partial X^m}\pi^{mi} + \frac{\partial \pi^{ki}}{\partial X^m}\pi^{mj} = 0 \tag{1}$$

Equation (1) is the Jacobi identity of the target space Poisson bracket,

$$\{F(X), G(X)\}_{PB} \equiv \tfrac{1}{2}\pi^{ij}(X)\frac{\partial F}{\partial X^i}\frac{\partial G}{\partial X^j}.$$

The gauge algebra is an open algebra,

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]X^i = \delta_{[\epsilon_1, \epsilon_2]}X^i,$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A_{\mu i} = \delta_{[\epsilon_1, \epsilon_2]}A_{\mu i} + \epsilon_{1j}\epsilon_{2k}\frac{\partial \pi^{jk}}{\partial X^i \partial X^l}(X)\frac{\delta S}{\delta A_{\mu l}}$$

§4. Batalin-Vilkovisky formalism

1, Replace the gauge parameter ϵ_i to the Grassmann odd FP ghost c_i with ghost number one.

A gauge transformation δ_ϵ is replaced to the BRST transformation s of degree one and define $sc_i = -\frac{1}{2}\frac{\partial \pi^{jk}}{\partial X^i} c_j c_k$, to satisfy $s^2 \approx 0$ (on-shell).

2, Introduce *antifields* $\Phi^* = (X^*, A^*, c^*)$ for each field $\Phi = (X, A, c)$ in BRST formalism such that $\mathrm{gh}\ \Phi + \mathrm{gh}\ \Phi^* = -1$.

3, Introduce the BV bracket (an odd Poisson bracket)

$$\{F, G\} \equiv \sum_{\Phi} \int_{\Sigma} \left(F \frac{\overleftarrow{\partial}}{\partial \Phi(\sigma)} \frac{\overrightarrow{\partial}}{\partial \Phi^*(\sigma')} G - F \frac{\overleftarrow{\partial}}{\partial \Phi^*(\sigma)} \frac{\overrightarrow{\partial}}{\partial \Phi(\sigma')} G \right) \delta^2(\sigma - \sigma')$$

4, Batalin-Vilkovisky (BV) action functional S_{BV} is determined by imposing the condition, $\{S_{BV}, S_{BV}\} = 0$ in off-shell, (the classical master equation), where S_{BV} is expanded by Φ^*,

$$S_{BV} = S + (-1)^{\mathrm{gh}\Phi} \int_{\Sigma} \Phi^* s \Phi + S_2(\Phi^{*2}) + S_3(\Phi^{*3}) + \ldots .$$

Solution

$$S_{BV} = \int_\Sigma \Big[A_i \wedge \mathrm{d}X^i + \tfrac{1}{2}\pi^{ij}(X) A_i \wedge A_j$$
$$-X^{+i}\pi^{ij}c_j + A^{+i} \wedge \left(\mathrm{d}c_i + \frac{\partial \pi^{jk}}{\partial X^i} A_j c_k\right) + \frac{1}{2}\frac{\partial \pi^{jk}}{\partial X^i} c^{+i} c_j c_k$$
$$-\frac{1}{4}\frac{\partial^2 \pi^{kl}}{\partial X^i \partial X^j} A^{+i} \wedge A^{+j} c_k c_l\Big],$$

where $A_i \equiv \mathrm{d}\sigma^\mu A_{\mu i}$, $A^{+i} \equiv \mathrm{d}\sigma^\mu \epsilon_{\mu\nu} A^{*\nu i}$, $X_i^+ \equiv \frac{1}{2}\mathrm{d}\sigma^\mu \wedge \mathrm{d}\sigma^\nu \epsilon_{\mu\nu} X_i^*$, $c^{+i} \equiv \frac{1}{2}\mathrm{d}\sigma^\mu \wedge \mathrm{d}\sigma^\nu \epsilon_{\mu\nu} c^{*i}$

5, The BRST transformation is defined by

$$sf[\Phi, \Phi^*] = \{S_{BV}, f[\Phi, \Phi^*]\}$$

6, Since $\{S_{BV}, S_{BV}\} = 0$,

i) S_{BV} is gauge invariant because $sS_{BV} = \{S_{BV}, S_{BV}\} = 0$,

ii) $s^2 = 0$ (off-shell) because $s^2F = \{S_{BV}, \{S_{BV}, F\}\} = \frac{1}{2}\{\{S_{BV}, S_{BV}\}, F\} = 0$.

§4-2. Quantum BV

The partition function

$$Z = \int_{\mathcal{L}} \mathcal{D}\Phi \; e^{\frac{i}{\hbar} S_{BVq}}$$

must be invariant under changing of the Lagrangian submanifold $\mathcal{L}' = \mathcal{L} + \delta\mathcal{L}$. The quantum master equation

$$-2i\hbar\Delta S_{BVq} + \{S_{BVq}, S_{BVq}\} = 0.$$

Here Δ is the BV Laplacian, $\Delta \equiv \int \frac{\partial}{\partial \Phi^I} \frac{\partial}{\partial \Phi^*_I}$.

§5. QP-manifolds (Differential graded symplectic manifolds)

Physics $\longrightarrow$ Mathematics

BV formalism $\longrightarrow$ QP-manifold (differential graded symplectic manifold)

1, fields + ghosts + antifields $\longrightarrow$ graded manifold

2, BV bracket $\longrightarrow$ graded symplectic form and odd Poisson bracket

3, BV action functional and the classical master equation $\longrightarrow$ Homological vector field and its Hamiltonian function

Definition 1. *The triple* $(\mathcal{M}, \omega, Q)$ *is called a QP-manifold of degree* n

1, $\mathcal{M}$: graded manifolds

A **graded manifold** $\mathcal{M} = (M, \mathcal{O}_M)$ on a smooth manifold M is a ringed space which structure sheaf $\mathcal{O}_M$ is $\boldsymbol{Z}$–graded commutative algebras over M, locally isomorphic to $C^\infty(U) \otimes S^\bullet(V)$, where U is a local chart on M, V is a graded vector space and $S^\bullet(V)$ is a free graded commutative ring on V.

Grading is called **degree**. We denote $\mathcal{O}_M = C^\infty(\mathcal{M})$.

2. ω: (P-structure) a graded symplectic form of degree n on $\mathcal{M}$.

3. Q: (Q-structure) (Homological vector field)
A graded vector field of degree $+1$ such that $Q^2 = 0$, and $\mathcal{L}_Q \omega = 0$.

Note: $(\mathcal{M}, Q)$ is called a *Q-manifold* if $Q^2 = 0$.

Note: A graded Poisson bracket $\{-,-\}$ of degree $-n$ is induced from ω.

$$
\begin{aligned}
&\{f, g\} = -(-1)^{(|f|-n)(|g|-n)}\{g, f\}, \\
&\{f, gh\} = \{f, g\}h + (-1)^{(|f|-n)|g|}g\{f, h\}, \\
&\{f, \{g, h\}\} = \{\{f, g\}, h\} + (-1)^{(|f|-n)(|g|-n)}\{g, \{f, h\}\}.
\end{aligned}
$$

Note: If degree $n \neq 0$, there exists a Hamiltonian function (a homological function) $\Theta \in C^\infty(\mathcal{M})$ of degree $n+1$ such that $Q(-) = \{\Theta, -\}$.

$Q^2 = 0$ is equal to the equation, $\{\Theta, \Theta\} = 0$.

BRST-BV formalism of gauge theory

A (classical) BRST-BV formalism is a QP-manifold of degree -1 on the mapping space of two graded manifolds.

BV of Poisson sigma model

$\mathcal{M} = \mathrm{Map}(T[1]\Sigma, T^*[1]M) \simeq T^*[-1]\mathrm{Map}(T[1]\Sigma, M)$

ω induce the BV bracket.

$\Theta = S_{BV}$ is the homological function.

§6. Geometry of Q(P)-manifolds

Lie algebras Let $\mathfrak{g}$ be a vector space.

1. $\mathcal{M} = T^*[2]\mathfrak{g}[1] \simeq \mathfrak{g}[1] \oplus \mathfrak{g}^*[1]$. Let c^a and b_a be odd coordinates of $\mathfrak{g}[1]$ and $\mathfrak{g}^*[1]$.

2. The symplectic form is $\omega = \delta c^a \wedge \delta b_a$ and $\{c^a, b_b\} = \delta^a_b$.

3. $\Theta = \frac{1}{2} C_{ab}^c c^a c^b b_c$. $\{\Theta, \Theta\} = 0$ is equivalent that C_{ab}^c is a structure constant of a Lie algebra with $[b_a, b_b] := C_{ab}^c b_c$.

$C^\infty(T^*[2]\mathfrak{g}[1]) \simeq \wedge^\bullet(\mathfrak{g} \oplus \mathfrak{g}^*)$ with Q is the Chevalley-Eilenberg complex of $\mathfrak{g}$.

§6-1. Q-manifold and QP-manifold of degree one

Shifted vector bundles $E[1]$ and Lie algebroids

Let E be a vector bundle over a smooth manifold M. Let (x^i, q^a) be a local coordinates on $E[1]$ of degree $(0, 1)$.

Let Q be a homological vector field of degree one such that $Q^2 = 0$. A general form is

$$Q \;=\; \rho_a^i(x) q^a \frac{\partial}{\partial x^i} - \frac{1}{2} C_{bc}^a(x) q^b q^c \frac{\partial}{\partial q^a}.$$

Proposition 1. *A Q-manifold $(E[1], Q)$ induces a Lie algebroid structure on E.*

Definition 2. *A Lie algebroid* $(E, \rho, [-,-])$ *is a vector bundle* E *over* M *with a bundle map* $\rho : E \to TM$ *called the anchor map, and a Lie bracket* $[-,-] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ *satisfying the Leibniz rule,*

$$[e_1, fe_2] \;\; = \;\; f[e_1, e_2] + \rho(e_1)f \cdot e_2,$$

where $e_i \in \Gamma(E)$ *and* $f \in C^\infty(M)$*.*

Define $\rho(e_a) := \rho_a^i(x)\partial_i$ and $[e_a, e_b] := C_{ab}^c(x)e_c$ for the basis e_a of E.

Example 1. [Lie algebras] *Let a manifold* M *be one point* $M = \{pt\}$*. Then it is a Lie algebra* $\mathfrak{g}$*.*

Example 2. [Tangent Lie algebroids] $E = TM$ *and* $\rho = \mathrm{id}$*,* $[-,-]$ *is a normal Lie bracket on the space of vector fields* $\mathfrak{X}(M)$*.*

Example 3. [Poisson Lie algebroid] *Let* (M, π) *be a Poisson manifold.*

A Poisson structure is defined by the bivector field $\pi = \frac{1}{2}\pi^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \in \Gamma(\wedge^2 TM)$ *with* $[\pi, \pi]_S = 0$*, where* $[-,-]_S$ *is the Schouten bracket.*

Then, T^*M is a Lie algebroid.

A bundle map $\pi^\sharp$ is defined as $\pi^\sharp : T^*M \to TM$ by $\langle \pi^\sharp(\alpha),\, \beta \rangle = \pi(\alpha, \beta)$ for all $\beta \in \Omega^1(M)$.

A Lie bracket on $\Omega^1(M)$ is given by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp(\alpha)}\beta - \mathcal{L}_{\pi^\sharp(\beta)}\alpha - \mathrm{d}(\pi(\alpha, \beta)),$$

where $\alpha, \beta \in \Omega^1(M)$.

Proposition 2. *A QP-manifold $(T^*[1]M, \omega, Q)$ induces a Poisson structure on M.*

§6-2. QP-manifold of degree two

1. A graded manifold is $\mathcal{M} = T^*[2]E[1] = (M, \mathcal{O}_M)$, where E is a vector bundle on M.

Assume a fiber metric $\langle -, - \rangle = k$ to identify E and E^*.

A local coordinate is (x^i, η^a) of degree $(0, 1)$ and the conjugate coordinate is $(\xi_i, k_{ab}\eta^b)$ of degree $(2, 1)$.

2. graded symplectic form of degree two

$\omega = \delta x^i \wedge \delta \xi_i + \frac{1}{2}\delta(k_{ab}\eta^a) \wedge \delta \eta^b$.

3. A general homological function of degree 3 is

$$\Theta = \rho^i{}_a(x)\xi_i\eta^a + \frac{1}{3!}H_{abc}(x)\eta^a\eta^b\eta^c.$$

and is imposed $\{\Theta, \Theta\} = 0$.

Structure sheaf $\mathcal{O}_M = C^\infty(\mathcal{M})$ is not described by a space of sections of a vector bundle.

We decompose $C^\infty(\mathcal{M}) = \sum_{i\geq 0} C_i(\mathcal{M})$, where $C_i(\mathcal{M})$ is the space of functions of degree i, and take $C_0(\mathcal{M}) \oplus C_1(\mathcal{M})$.

Theorem 1. *Roytenberg '99*

A QP manifold of degree 2 induces a $\boldsymbol{Courant\ algebroid}$ *structure on E.*

Courant sigma model

We can construct a sigma model with a Courant algebroid structure.

NI '02, Roytenberg '06

Definition 3. [Courant algebroids] *Liu, Weinstein, Xu '97, Kosmann-Schwarzbach '07*

Let E be a vector bundle over M equipped with a pseudo-Euclidean inner product $\langle -, - \rangle$, a bundle map $\rho : E \longrightarrow TM$ and a binary bracket $[-,-]_D$ on $\Gamma(E)$. The bundle is called the **Courant algebroid** *if three conditions are satisfied,*

$$
\begin{aligned}
[e_1,[e_2,e_3]_D]_D &= [[e_1,e_2]_D,e_3]_D + [e_2,[e_1,e_3]_D]_D, \\
\rho(e_1)\langle e_2,\, e_3\rangle &= \langle [e_1,e_2]_D,\, e_3\rangle + \langle e_2,\, [e_1,e_3]_D\rangle, \\
\rho(e_1)\langle e_2,\, e_3\rangle &= \langle e_1,\, [e_2,e_3]_D + [e_3,e_2]_D\rangle,
\end{aligned}
$$

where $e_1, e_2, e_3 \in \Gamma(E)$.

Derived bracket construction of Courant algebroids

$C_0(\mathcal{M})$ and $C_1(\mathcal{M})$ make a closed algebra by the derived bracket $\{\{-, \Theta\}, -\}$. (Count degree!)

$$C_0(\mathcal{M}) \simeq C^\infty(M), \quad f(x).$$
$$C_1(\mathcal{M}) \simeq \Gamma(E), \quad \alpha_a(x)\eta^a \in C_1(\mathcal{M}) \simeq e = \alpha_a(x)e^a \in \Gamma(E).$$

The operations on E are defined by Poisson brackets and **derived brackets**.

For $f, g \in C_0(\mathcal{M})$, $e, e_1, e_2 \in C_1(\mathcal{M})$, Poisson brackets

$C_0 \times C_0, \quad 0 = \{f, g\}$

$C_1 \times C_0, \quad 0 = \{e, f\}$

$C_1 \times C_1, \quad \langle e_1,\, e_2 \rangle = \{e_1, e_2\}$ (inner product)

Derived brackets

$C_0 \times C_0 \to 0, \quad 0 = \{\{f, \Theta\}, g\}$

$C_1 \times C_0 \to C_0, \quad \rho(e)f = -\{\{e, \Theta\}, f\}$ (anchor map)

$C_1 \times C_1 \to C_1, \quad [e_1, e_2]_D = -\{\{e_1, \Theta\}, e_2\}$ (Dorfman bracket)

Example 4. *For general n, we call the structure on the vector bundle E induced by the derived brackets on the corresponding QP manifold of degree n, a* ***Lie n-algebroid****.*

A ***Lie 1-algebroid*** *is a Poisson-Lie algebroid.*

A ***Lie 2-algebroid*** *is a Courant algebroid.*

A corresponding global object to a Lie n-algebroid is called a Lie n-groupoid.

Example: bc-$\beta\gamma$ system in superstring theory

It is a QP-manifold of degree two, $T^*[2]E[1]$.

$(b^a, c_a, \beta_i, \gamma^i)$ are local coordinates of degree $(1, 1, 2, 0)$. Graded Poisson brackets $\{b^a, c_b\} = \delta^a_b$, $\{\beta_i, \gamma^j\} = \delta_i^j$ are induced from ω.

Local coordinate transformations on a QP-manifold of degree 2 are equal to canonical transformations on the bc-$\beta\gamma$ system,

$$\begin{aligned} \gamma'^i &= \gamma'^i(\gamma), \qquad b'^a = M_b^a(\gamma) b^b, \qquad c'_a = M_a^b(\gamma) c_b, \\ \beta'_i &= \frac{\partial \gamma^j}{\partial \gamma'^i} \beta_j + \frac{1}{2} M_b^c \frac{\partial M_c^d}{\partial \gamma'^i} b^b c_d. \end{aligned}$$

§7. Geometric constructions of BV

NI-Strobl '21, Chatzistavrakidis-NI-Šimunić '22, Chatzistavrakidis-NI-Jonke '24

The BV action functional is constructed using geometric quantities of Lie and higher algebroids.

Two differentials in Lie algebroids

$$\mathrm{d} : \Gamma(\wedge^l T^*M) \to \Gamma(\wedge^{l+1} T^*M),$$
$$^E\mathrm{d} : \Gamma(\wedge^m E^*) \to \Gamma(\wedge^{m+1} E^*).$$

d is the de Rham differential.

E-differential (Lie algebroid differentials)

$\Gamma(\wedge^\bullet E^*)$ is the space of E-differential forms.

Definition 4. *For $\alpha \in \Gamma(\wedge^m E^*)$ and $e_i \in \Gamma(E)$, an E-differential ${}^E\mathrm{d} : \Gamma(\wedge^m E^*) \to \Gamma(\wedge^{m+1} E^*)$ such that $({}^E\mathrm{d})^2 = 0$ is defined by*

$$
\begin{aligned}
{}^E\mathrm{d}\alpha(e_1, \ldots, e_{m+1}) &= \sum_{i=1}^{m+1} (-1)^{i-1} \rho(e_i) \alpha(e_1, \ldots, \check{e}_i, \ldots, e_{m+1}) \\
&\quad + \sum_{1 \le i < j \le m+1} (-1)^{i+j} \alpha([e_i, e_j], e_1, \ldots, \check{e}_i, \ldots, \check{e}_j, \ldots, e_{m+1}).
\end{aligned}
$$

Two connections

Definition 5. *A* $connection$ *on a vector bundle* E' *is a* $\boldsymbol{R}$*-linear map* $\nabla : \Gamma(E') \to \Gamma(E' \otimes T^*M)$ *satisfying*

$$\nabla_v(fe') = f\nabla_v e' + (vf)e',$$

for $v \in \Gamma(TM)$*,* $e' \in \Gamma(E')$ *and* $f \in C^\infty(M)$*.*

Definition 6. *An* E*-connection* *on a vector bundle* E' *with respect to a Lie algebroid* E *is a* $\boldsymbol{R}$*-linear map* ${}^E\nabla : \Gamma(E') \to \Gamma(E' \otimes E^*)$ *satisfying*

$${}^E\nabla_e(fe') = f\,{}^E\nabla_e e' + (\rho(e)f)e',$$

for $e \in \Gamma(E)$*,* $e' \in \Gamma(E')$ *and* $f \in C^\infty(M)$*.*

For a given (normal) vector bundle connection $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ on E, an E-connection called the *basic E-connection* on TM, ${}^E\nabla : \Gamma(TM) \to \Gamma(TM \otimes E^*)$ is defined by

$$ {}^E\nabla_e v \;\; := \;\; \mathcal{L}_{\rho(e)} v + \rho(\nabla_v e) = [\rho(e), v] + \rho(\nabla_v e). $$

The *basic E-connection* on E, ${}^E\nabla : \Gamma(E) \to \Gamma(E \otimes E^*)$ is defined by

$$ {}^E\nabla_e e' \;\; := \;\; \nabla_{\rho(e')} e + [e, e'], $$

for $e, e' \in \Gamma(E)$.

Another torsions and curvatures

Let $e, e' \in \Gamma(E)$, $v, v' \in \mathfrak{X}(M)$. The *$E$-torsion* and the *$E$-curvature*,

$$T(e,e') = \nabla_{\rho(e)}e' - \nabla_{\rho(e')}e + [e,e'] \in \Gamma(E \otimes \wedge^2 E^*)$$
$${}^E R(e,e') = [{}^E\nabla_e, {}^E\nabla'_e] - {}^E\nabla_{[e,e']} \in \Gamma(\wedge^2 E^* \otimes E \otimes E^*)$$

The *basic curvature*, $\mathcal{S} \in \Gamma(T^*M \otimes E \otimes \wedge^2 E^*)$ Blaom '06

$$\mathcal{S}(e,e')(v) \;=\; \nabla_v[e,e'] - [\nabla_v e, e'] - [e, \nabla_v e'] - \nabla_{{}^E\nabla_{e'}v}e + \nabla_{{}^E\nabla_e v}e'$$

Geometric form of BV action functional

The Poisson sigma model in two dimensions,

$$S_{BV} = \int_\Sigma \big[\langle A, \mathrm{d}_{\mathcal{X}} X \rangle + \tfrac{1}{2} (\pi \circ X)(A, A)$$
$$+ \langle A^+, \nabla c - (T \circ X)(A, c) \rangle - (\pi \circ X)(X^+, c)$$
$$- \tfrac{1}{2} \langle c^+, (T \circ X)(c, c) \rangle + \tfrac{1}{4} \langle A^+, (\mathcal{S} \circ X)(A^+, c, c) \rangle \big] \, .$$

Note: S_{BV} does not depend on the connection ∇ (and ${}^E\nabla$), but each term does.

Other results

Geometric BV formalism of a *(pre-)Courant sigma model* is analyzed.

The BV action functional is constructed using the E-torsion, the basic curvature of the (pre-)Courant algebroid.

Chatzistavrakidis-NI-Šimunić, '22, Chatzistavrakidis-NI-Jonke, '23

§Further applications

- Σ_{n+1}: higher dimensional worldvolume, higher Lie n-algebroid, applications to n-brane geometry, M-theory, T-duality, U-duality

- Geometry of gauged nonlinear sigma models and generalizations of momentum maps on multisymplectic manifolds NI '18, Hirota-NI '22

- Construction of current algebras induced from QP manifolds NI-Koizumi '13, NI-Xu '14, Arvanitakis '21, Hayami '23

- Applications to mathematics, noncommutative geometry, deformation quantization, geometry of algebroids, connections, curvatures, momentum maps, localizations

§. Outlook

• The constraint algebra in the general relativity in four dimensions is a Lie algebroid. Blohmann-Fernandes-Weinstein '13

• higher degree QP-manifolds and analysis of higher dimensional theories

• Quantization (deformation quantizations, path integral quantizations)

Thank you for your attention!