Integrable system related to the restricted Grassmannian on partial isometries

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Restricted Grassmannian

• fix an orthogonal decomposition (called polarization) of the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

onto infinite dimensional Hilbert subspaces $\mathcal{H}_\pm$.

• P_+, P_-: the orthogonal projectors onto $\mathcal{H}_+$ and $\mathcal{H}_-$

• block decomposition of an operator A acting on $\mathcal{H}$:

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

Definition

The **restricted Grassmannian** $\mathrm{Gr}_{\mathrm{res}}$ is defined as a set of Hilbert subspaces $W \subset \mathcal{H}$ such that:

- the orthogonal projection $p_+ : W \to \mathcal{H}_+$ is a Fredholm operator;
- the orthogonal projection $p_- : W \to \mathcal{H}_-$ is a Hilbert–Schmidt operator.

• identify the Hilbert subspace W with a projector P_W onto this subspace.

Proposition

$$W \in \mathrm{Gr}_{\mathrm{res}} \Longleftrightarrow P_W - P_+ \in L^2$$

• Banach Lie group: unitary restricted group $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ acting transitively on $\mathrm{Gr}_{\mathrm{res}}$:

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}) := \{u \in \mathrm{U}(\mathcal{H}) \mid [u, P_+] \in L^2\}$$

• Banach Lie group: unitary restricted group $\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ acting transitively on $\mathrm{Gr}_{\mathrm{res}}$:

$$\mathrm{U}_{\mathrm{res}}(\mathcal{H}) := \{u \in \mathrm{U}(\mathcal{H}) \mid [u, P_+] \in L^2\}$$

• its Banach Lie algebra

$$\mathfrak{u}_{\mathrm{res}}(\mathcal{H}) := \{A \in \mathfrak{u}(\mathcal{H}) \mid [A, P_+] \in L^2\}$$

• $\mathrm{Gr}_{\mathrm{res}}$ can be seen as a smooth homogenous space $\mathrm{U}_{\mathrm{res}}(\mathcal{H})/(U_+ \times U_-)$

Banach Lie–Poisson space

• $\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H}) := \{\mu \in L^\infty(\mathcal{H}) \mid \mu^* = -\mu, \mu_{-+}, \mu_{+-} \in L^2, \mu_{++}, \mu_{--} \in L^1\}$ is a predual space to $\mathfrak{u}_{\mathrm{res}}(\mathcal{H})$

$$\langle \mu \, ; \, A \rangle := \mathrm{Tr}_{\mathrm{res}}(\mu A),$$

where $\mathrm{Tr}_{\mathrm{res}}$ is the **restricted trace** defined on $\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$ by

$$\mathrm{Tr}_{\mathrm{res}}\, \mu := \mathrm{Tr}(\mu_{++} + \mu_{--})$$

Banach Lie–Poisson space

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$$\langle \mu \; ; \; A\rangle := \mathrm{Tr}_{\rm res}(\mu A),$$

where $\mathrm{Tr}_{\rm res}$ is the **restricted trace** defined on $\mathfrak{u}^1_{\rm res}(\mathcal{H})$ by

$$\mathrm{Tr}_{\rm res}\,\mu := \mathrm{Tr}(\mu_{++} + \mu_{--})$$

• $\mathrm{Tr}_{\rm res}$ is defined on a larger domain than $L^1(\mathcal{H})$ and it coincides with the standard trace Tr there.

• It is a Banach Lie–Poisson space with respect to the Poisson bracket

$$\{f, g\}_0(\mu, \gamma) = \mathrm{Tr}_{\rm res}\left(\mu[Df(\mu), Dg(\mu)]\right)$$

Central extensions

- Cocycle — Schwinger term

$$s(X,Y) = \mathrm{Tr}(X_{+-}Y_{-+} - Y_{+-}X_{-+})$$

$X, Y \in \mathfrak{u}_{\mathrm{res}}(\mathcal{H})$

$$\widetilde{\mathfrak{u}}_{\mathrm{res}}(\mathcal{H}) := \mathfrak{u}_{\mathrm{res}}(\mathcal{H}) \oplus i\mathbb{R}$$

$$[(X,\gamma),(Y,\gamma')] = \big([X,Y], -s(X,Y)\big)$$

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• predual of $\widetilde{\mathfrak{u}}_{\mathrm{res}}(\mathcal{H})$

$$\widetilde{\mathfrak{u}}^{1}_{\mathrm{res}}(\mathcal{H}) := \mathfrak{u}^{1}_{\mathrm{res}}(\mathcal{H}) \oplus i\mathbb{R}$$

$$\langle(\mu,\gamma)\ ;\ (X,\gamma)\rangle_{\sim} = \mathrm{Tr}_{\mathrm{res}}(\mu X) + \gamma\lambda,$$

$\mu \in \mathfrak{u}^{1}_{\mathrm{res}}(\mathcal{H}),\ X \in \mathfrak{u}_{\mathrm{res}}(\mathcal{H}),\ \gamma,\lambda \in i\mathbb{R}.$

Pencil of Poisson brackets

• Poisson bracket on $\widetilde{\mathfrak{u}}^1_{\mathrm{res}}(\mathcal{H})$

$$\{F,G\}(\mu,\gamma) = \langle(\mu,\gamma)\ ;\ [DF(\mu,\gamma), DG(\mu,\gamma)]\rangle_{\sim} =$$

$$= \mathrm{Tr}_{\mathrm{res}}\left(\mu[D_1F(\mu,\gamma), D_1G(\mu,\gamma)]\right) - \gamma s(D_1F(\mu,\gamma), D_1G(\mu,\gamma)),$$

where D_1 is the derivation with respect to the first argument of functions $F, G \in C^\infty(\widetilde{\mathfrak{u}}^1_{\mathrm{res}}(\mathcal{H}))$.

• The extension is central, there is no derivative with respect to γ in this Poisson bracket. We consider the variable γ as a parameter and obtain a pencil of Poisson brackets on $\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$

$$\{f,g\}_\gamma(\mu) = \{f,g\}_0(\mu) - \gamma\{f,g\}_s(\mu)$$

for $f, g \in C^\infty(\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H}))$

$$\{f,g\}_s(\mu) = s(Df(\mu), Dg(\mu)) = \mathrm{Tr}\left(Df(\mu)[Dg(\mu), P_+]\right)$$

Formal remark

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Forgetting about convergence one might be tempted to write

$$\{f, g\}_s(\mu) = -\gamma\,\mathrm{Tr}_{\mathrm{res}}\left(P_+[DF(\mu), DG(\mu)]\right).$$

It looks just like a Mishchenko–Fomenko “frozen bracket”.

Regretfully, that expression doesn’t make sense, but allows us to guess the Casimirs.

Casimirs

$$I^n_\gamma(\mu) := i^{n+1}\,\mathrm{Tr}_{\mathrm{res}}\left((\mu-\gamma P_+)^{n+1} - (-\gamma)^n(\mu-\gamma P_+)\right)$$

$$I^n_\gamma(\mu) = i^{n+1}\sum_{k=0}^{n}(-\gamma)^k\,\mathrm{Tr}_{\mathrm{res}}\,W^{n+1}_k(\mu) + i^{n+1}(-\gamma)^n\,\mathrm{Tr}_{\mathrm{res}}\,\mu$$

where

$$(\mu+\gamma P_+)^n = \sum_{k=0}^{n}\gamma^k W^n_k(\mu)$$

Hamiltonians in involution on $\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$

$$h^n_k(\mu) = i^{n+1}\,\mathrm{Tr}_{\mathrm{res}}\,W^{n+1}_k(\mu), \quad 0 \leqslant k \leqslant n$$

$$\{h^n_k, h^m_l\}_0 = \{h^n_k, h^m_l\}_s = 0$$

Magri chain

$$\{h_k^n, \cdot\}_0 = \{h_{k+1}^n, \cdot\}_s$$

$$\frac{\partial}{\partial \tau_k^n}\mu = -i^{n+1}(n+1)[\mu, W_k^n(\mu)]$$

or equivalently

$$\frac{\partial}{\partial \tau_k^n}\mu = i^{n+1}(n+1)[P_+, W_{k-1}^n(\mu)],$$

$$
\begin{aligned}
W_n^n &= P_+ \\
W_{n-1}^n &= \mu P_+ + P_+\mu + (n-2)P_+\mu P_+ \qquad\qquad n \geqslant 2 \\
W_{n-2}^n &= \mu^2 P_+ + \mu P_+\mu + P_+\mu^2 + \\
&\quad + (n-3)\left(P_+\mu^2 P_+ + P_+\mu P_+\mu + \mu P_+\mu P_+\right) + \\
&\quad + \frac{(n-3)(n-4)}{2} P_+\mu P_+\mu P_+ \qquad\qquad n \geqslant 4 \\
&\vdots \\
W_1^n &= P_+\mu^{n-1} + \mu P_+\mu^{n-2} + \ldots + \mu^{n-1}P_+ \\
W_0^n &= \mu^n
\end{aligned}
$$

• homogeneous polynomials

$$H_k^n(\mu) := \sum_{\substack{i_0, i_1, \dots i_n \in \{0,1\} \\ i_0 + \dots + i_n = k}} P_+^{i_0} \mu P_+^{i_1} \mu \dots \mu P_+^{i_n}$$

of the degree $n \in \mathbb{N}$ in the operator variable $\mu \in \mathfrak{u}^1_{\mathrm{res}}$ and degree k in P_+, where $k \leqslant n+1$.

• hierarchy of commuting equations (Lax form)

$$\frac{\partial}{\partial t_k^n} \mu = i^{n+1} [\mu, H_k^n(\mu)]$$

Proposition

The diagonal blocks μ_{++} and μ_{--} are constant

$$\frac{\partial}{\partial t_k^n}\mu_{++} = 0 \qquad \frac{\partial}{\partial t_k^n}\mu_{--} = 0$$

Proposition

The diagonal blocks μ_{++} *and* μ_{--} *are constant*

$$\frac{\partial}{\partial t_k^n}\mu_{++} = 0 \qquad \frac{\partial}{\partial t_k^n}\mu_{--} = 0$$

Proof.

Follows from considering symplectic leaves of $\{\cdot,\cdot\}_s$ or computing the momentum map of the action of the group $\mathrm{U}(\mathcal{H}_+)\times\mathrm{U}(\mathcal{H}_-)\subset\mathrm{U}_{\mathrm{res}}(\mathcal{H})$ on the Poisson manifold $(\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H}),\{\cdot,\cdot\}_0)$:

$$J(\mu) = p_D(\mu),$$

where p_D is the projection onto block-diagonal part

$$p_D(\mu) = P_+\mu P_+ + P_-\mu P_- \in \mathfrak{u}^1(\mathcal{H}_+)\oplus\mathfrak{u}^1(\mathcal{H}_-)$$

□

Proposition

In the case $\mu_{++} = 0$ *the modulus* $|\mu_{-+}|$ *is constant along the bihamiltonian flows for all* t_k^n, $n \in \mathbb{N}$, $k \leqslant n+1$.

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Proof.

For $k = 1$ one can compute that

$$\frac{\partial}{\partial t_1^n}\left(\mu_{+-}\mu_{-+}\right) = i^{n+1}[(\mu^{n+1})_{++}, \mu_{++}]$$

Now if we assume that the block $\mu_{++} = 0$ for all t_1^n, we see that $|\mu_{-+}|$ is constant.

For $k > 1$ the computations are a bit more involved but still straightforward. □

Consider the polar decomposition of $\mu_{-+} = uB$.

Proposition

Assume that $\mu_{++} = 0$ and $|\mu_{-+}|$ is partially invertible. The equations for the evolution of the partial isometry u assume the form

$$\frac{\partial}{\partial t_k^n} u = i^{n+1} (\mu H_{k-1}^{n-1})_{--} u$$

for $n \in \mathbb{N}$, $k \leqslant n + 1$.

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for $n \in \mathbb{N}$, $k \leqslant n+1$.

For $k = 1$

$$\frac{\partial}{\partial t_1^n} u = i^{n+1} (\mu^n)_{--} u.$$

$$\mu = \begin{pmatrix} 0 & -Bu^* \\ uB & D \end{pmatrix}$$

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$$\frac{\partial}{\partial t_1^1} u = -Du$$

$$\frac{\partial}{\partial t_1^2} u = i(uB^2 - D^2 u)$$

$$\frac{\partial}{\partial t_1^3} u = -DuB^2 - uB^2 u^* Du + D^3 u$$

$$\frac{\partial}{\partial t_2^3} u = -DuB^2 - uB^2 u^* Du$$

$$\frac{\partial}{\partial t_2^4} u = i(2uB^4 - D^2 uB^2 - uB^2 u^* D^2 u - DuB^2 u^* Du)$$

Partial isometry $u : \mathcal{H}_+ \to \mathcal{H}_-$ evolves in such a way that its initial space is constant and final space evolves with time.

$$\frac{\partial}{\partial t_1^n}(uu^*) = -i^{n+1}[uu^*, (\mu^n)_{--}]$$

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Remark

The partial isometry u can be extended trivially to a partial isometry in $\mathcal{H}$. In this way we obtain differential equations on the Banach Lie groupoid of partial isometries $\mathcal{U}(\mathcal{H})$ generating a flow on $s^{-1}((\ker B)^{\perp}) \cap t^{-1}(\mathrm{Gr}(\mathcal{H}_-)) \subset \mathcal{U}(\mathcal{H})$, where $\mathrm{Gr}(\mathcal{H}_-)$ is the Grassmannian of all closed subspaces of $\mathcal{H}_-$.

EXAMPLE: RANK $u = 1$

$$B = b|e_1\rangle\langle e_1|, \qquad \mathbb{R} \ni b > 0$$

$$u = |\psi\rangle\langle e_1|, \qquad \psi \in \mathcal{H}_-, \|\psi\| = 1$$

$$D = \sum d_i|f_i\rangle\langle f_i|, \qquad d_i \in i\mathbb{R}$$

$$\psi = \sum_{i=1}^{\infty} \alpha_i f_i$$

PROPOSITION

Evolution of the coefficients $\alpha_1, \alpha_2, \ldots$ of the vector ψ:

$$\frac{\partial}{\partial t_k^n}\alpha_j = i f_{j,k}^n(|\alpha_1|^2, |\alpha_2|^2, \ldots)\alpha_j,$$

where $f_{j,k}^n$ are smooth real-valued functions depending on the eigenvalues of the matrices b and d_i.

$$\alpha_j = r_j e^{i\varphi_j}$$

$$\begin{cases} \frac{\partial}{\partial t_k^n} r_j = 0 \\ \frac{\partial}{\partial t_k^n} \varphi_j = f_{j,k}^n(r_1^2, \dots, r_M^2) \end{cases}$$

Theorem

The solution for the case of partial isometries of rank one is the following

$$\alpha_j(t_1^1, t_1^2, t_2^2, \dots) = \alpha_j^0 \exp\left(i \sum_{n,k \leqslant n/2+1} f_{j,k}^n(\left|\alpha_1^0\right|^2, \dots, \left|\alpha_M^0\right|^2) t_k^n \right),$$

where $\alpha_j^0 \in \mathbb{C}$ *are the initial values.*

Restricted Grassmannian as a coadjoint orbit

$$\mathrm{Ad}^*_\Gamma(\mu,\gamma) = g^{-1}\mu g + \gamma(P_+ - g^{-1}P_+ g),$$

where $\Gamma \in \widetilde{\mathrm{U}}_{\mathrm{res}}(\mathcal{H})$ projects down to $g \in \mathrm{U}_{\mathrm{res}}(\mathcal{H})$.

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Diffeomorphism

$$\Phi_\gamma : \mathrm{Gr}_{\mathrm{res}} \ni W \longrightarrow \mu = \gamma(P_W - P_+) \in \mathcal{O}_{(0,\gamma)} \subset \mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$$

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Diffeomorphism

$$\Phi_\gamma : \mathrm{Gr}_{\mathrm{res}} \ni W \longrightarrow \mu = \gamma(P_W - P_+) \in \mathcal{O}_{(0,\gamma)} \subset \mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$$

Proposition

An element $\mu \in \mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$ belongs to the coadjoint orbit $\mathcal{O}_{(0,\gamma)}$ if and only if $\frac{1}{\gamma}\mu + P_+$ is an orthogonal projection.

$$\Omega_{\mathcal{H}_+} = \{W \in \mathrm{Gr}_{\mathrm{res}} \mid P_{W++} \text{ is invertible in } \mathcal{H}_+\}$$

An inverse of the chart $\varphi_{\mathcal{H}_+}$ on $\Omega_{\_H_+} \subset \mathrm{Gr}_{\mathrm{res}}$ is

$$\varphi^{-1}_{\mathcal{H}_+}(A) = \Gamma(A),$$

where $\Gamma(A)$ is the graph of an operator.

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$$\varphi_{\mathcal{H}_+}^{-1}(A) = \Gamma(A),$$

where $\Gamma(A)$ is the graph of an operator.

Composing the chart $\varphi_{\mathcal{H}_+}^{-1}$ with the diffeomorphism Φ_γ one obtains a parametrization of the restricted Grassmannian realized as a coadjoint orbit inside $\mathfrak{u}^1_{\mathrm{res}}(\mathcal{H})$:

$$\Phi_\gamma \circ \varphi_{\mathcal{H}_+}^{-1}(A) = \gamma \begin{pmatrix} (1 + A^*A)^{-1} - 1 & (1 + A^*A)^{-1}A^* \\ A(1 + A^*A)^{-1} & A(1 + A^*A)^{-1}A^* \end{pmatrix},$$

where $A \in L^2(\mathcal{H}_+, \mathcal{H}_-)$.

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PROPOSITION

For initial conditions in the affine coadjoint orbit $\mathcal{O}_{(0,\gamma)}$, the equations are linear when expressed in the chart $\varphi_{\mathcal{H}_+}$.

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Proof.

$p := \frac{1}{\gamma}\mu + P_+$ and $p^2 = p$ implies

$$\mu^2 = \gamma(\mu - \mu P_+ - P_+\mu) = \gamma(\mu_{--} - \mu_{++}).$$

μ^2 is constant and block diagonal. Moreover:

$$\begin{cases} \mu_{++}\mu_{+-} & = & -\mu_{+-}\mu_{--} \\ \mu_{-+}\mu_{++} & = & -\mu_{--}\mu_{-+} \\ \mu_{-+}\mu_{+-} & = & \text{const} \\ \mu_{+-}\mu_{-+} & = & \text{const} \end{cases}$$

□