A new look at Lie algebras

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Introduction

There is a well-known isomorphism between special orthogonal Lie algebra $\mathfrak{so}(3)$ and $\mathbb{R}^3$. For the first structure, the Lie bracket is given by the matrix commutator $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{so}(3)$, and for the second by the cross product $\times$ for vectors from $\mathbb{R}^3$. The mapping

$$X = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \longmapsto v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

gives this isomorphism $(\mathfrak{so}(3), [\cdot, \cdot]) \cong \left(\mathbb{R}^3, \times\right)$.
The main goal of the paper is to show that one can construct a similar isomorphism for any Lie algebra. We will show that Lie algebras have a lot in common with linear maps, and more precisely with linear maps with a fixed eigenvector.

Lie algebra

A Lie algebra is vector space over a field $\mathbb{R}$ equipped with Lie bracket $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ with is a bilinear, antisymmetric map, which satisfies the Jacobi identity

$$[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$$

for all $x, y, z \in \mathfrak{g}$.

The basic ingredient is a pair (F, v) consisting of a linear mapping $F \in End(V)$ with an eigenvector v. This pair allows to build a Lie bracket on a dual space to a linear space V.
In our considerations, we will restrict ourselves to the linear space V over a field $\mathbb{R}$. It means that we will analyze in detail only real Lie algebras. However, we want to emphasize that the presented formulas also work for vector spaces over the field of complex numbers.

Eigenvalue problem

We present some constructions of a Lie bracket on a space V^* having a pair: linear mapping and its eigenvector. A pair (F, v) gives a Lie bracket on a dual space V^*:

Theorem

If V is a vector space, $F : V \longrightarrow V$ is a linear map and $v \in V$ is an eigenvector of the map F, then $(V^*, [\cdot,\cdot]_{(F,v)})$, is a Lie algebra, where the Lie bracket is given by

$$[\psi, \phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi)$$

for $\psi, \phi \in V^*$.

$V \simeq V^* \simeq \mathbb{R}^N$

We can identify V and V^* with $\mathbb{R}^N$ with the canonical basis $\{e_1, e_2, \dots, e_N\}$ (i.e. $V \simeq V^* \simeq \mathbb{R}^N$), so that the pairing between V and V^* is given by the scalar product. Then the Lie bracket can be rewritten in the form

$$[u, w]_{(F,v)} = \langle w|v\rangle F^T u - \langle u|v\rangle F^T w \quad \text{for } u, w \in \mathbb{R}^N,$$

where $\langle\cdot|\cdot\rangle$ is the scalar product in $\mathbb{R}^N$.

Theorem

Let $[\cdot,\cdot]_{(F,v)}$ *be given by*

$$[\psi,\phi]_{(F,v)} = \phi(v)F^*(\psi) - \psi(v)F^*(\phi),$$

then the Lie algebra $(V^*, [\cdot,\cdot]_{(F,v)})$ *is solvable.*

Proof.

We say that a linear subspace $\mathfrak{h}$ is an ideal of a Lie algebra $\mathfrak{g}$ when $[\mathfrak{g},\mathfrak{h}] \subseteq \mathfrak{h}$. Of course the set $[\mathfrak{h},\mathfrak{h}]$ is also an ideal. Then we define a sequence of ideals (the derived series $\mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(i)} \supseteq \dots$)

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}],\ \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)},\mathfrak{g}^{(1)}], \dots, \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)},\mathfrak{g}^{(i-1)}], \dots$$

A Lie algebra $\mathfrak{g}$ is called solvable if, for some positive integer i, $\mathfrak{g}^{(i)} = 0$.
In this case we get $\mathfrak{g}^{(2)} = [[\mathfrak{g},\mathfrak{g}]_{(F,v)}, [\mathfrak{g},\mathfrak{g}]_{(F,v)}]_{(F,v)} = 0$. □

In addition, if we introduce the following sequence of ideals (the lower central series $\mathfrak{g}_{(0)} \supseteq \mathfrak{g}_{(1)} \supseteq \cdots \supseteq \mathfrak{g}_{(i)} \supseteq \ldots$)

$$\mathfrak{g}_{(0)} = \mathfrak{g},\ \mathfrak{g}_{(1)} = [\mathfrak{g}_{(0)}, \mathfrak{g}],\ \mathfrak{g}_{(2)} = [\mathfrak{g}_{(1)}, \mathfrak{g}], \ldots, \mathfrak{g}_{(i)} = [\mathfrak{g}_{(i-1)}, \mathfrak{g}], \ldots,$$

we say that algebra $\mathfrak{g}$ is called nilpotent if the lower central series terminates $\mathfrak{g}_{(i)} = 0$ for some $i \in \mathbb{N}$. Obviously, a nilpotent Lie algebra is also solvable.

Theorem

If F is a nilpotent operator, then $(V^*, [\cdot,\cdot]_{(F,v)})$ is a nilpotent Lie algebra.

Lie algebra generalized $ax + b$-group

If we look at this bracket we notice that this is a structure of a Lie bracket for a Lie algebra generalized $ax + b$-group

$$[(w_1, t_1), (w_2, t_2)] = (t_1 D w_2 - t_2 D w_1, 0),$$

where $V = W \ltimes \mathbb{R}$, W is $N-1$-dimensional linear space, $w_1, w_2 \in W$, $t_1, t_2 \in \mathbb{R}$ and D is established endomorphism $End(W)$. Identification is given by association $V \cong V^* \cong \mathbb{R}^N$ and putting $\psi = (w_1, t_1), \phi = (w_2, t_2)$, $v = e_N$, $F^* = \left(\begin{array}{c|c} -D & 0 \\ \hline 0 & 0 \end{array}\right)$

$$[\psi, \phi]_{(F,v)} = \phi(v) F^*(\psi) - \psi(v) F^*(\phi),$$

I. Beltiță, D. Beltiță, *Quasidiagonality of C^*-algebras of solvable Lie groups*, Integr. Equ. Oper. Theory, **90**:5, 2018.

We show that these solvable algebras are the basic bricks of the construction of all other Lie algebras.

The linear combination of Lie brackets $[\cdot,\cdot]_{(F,v)}$, $[\cdot,\cdot]_{(G,w)}$ gives a Lie bracket

Theorem

Let V be a vector space over $\mathbb{R}$. If $F, G \in End(V)$, $v, w \in V$ are such that:

- *v is an eigenvector of the map F,*
- *w is an eigenvector of the map G,*
- *the following condition is true*

$$v \wedge w \wedge [F,G]^* + w \wedge Gv \wedge F^* + v \wedge Fw \wedge G^* = 0.$$

Then $(V^*, [\cdot,\cdot]^{\lambda}_{F,v,G,w})$, where

$$[\psi,\phi]^{\lambda}_{(F,v),(G,w)} = [\psi,\phi]_{(F,v)} + \lambda[\psi,\phi]_{(G,w)}$$

is a Lie algebra for every $\lambda \in \mathbb{R}$.

Example – the three-dimensional Lie algebras

Let us take $V = \mathbb{R}^3$ with the standard basis $\{e_1, e_2, e_3\}$. We will show how to easily connect three-dimensional real Lie algebras with the corresponding linear mappings and their eigenvectors. We will restrict ourselves to the eigenvector $v = (0,0,1)^\top$. Lie brackets will be defined in the space $V^* = \left(\mathbb{R}^3\right)^\top$ with the dual base $\{e_1^*, e_2^*, e_3^*\}$.

Patera, J., Sharp, R.T., Winternitz, P., Zassenhaus, H.: Invariants of real low dimension Lie algebras. J. Math. Phys. 17. 986 (1976)

If we take

$$F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, we obtain the Lie bracket of the form

$$[\psi, \phi]_{(F,v)} = \lambda_1 \left(\psi_1\phi_3 - \psi_3\phi_1\right) e_1^* + \lambda_2 \left(\psi_2\phi_3 - \psi_3\phi_2\right) e_2^*,$$

where $\psi = \psi_1 e_1^* + \psi_2 e_2^* + \psi_3 e_3^*$ and $\phi = \phi_1 e_1^* + \phi_2 e_2^* + \phi_3 e_3^*$. The commutator rules are following

$$[e_1^*, e_2^*]_{(F,v)} = 0, \quad [e_1^*, e_3^*]_{(F,v)} = \lambda_1 e_1^*, \quad [e_2^*, e_3^*]_{(F,v)} = \lambda_2 e_2^*.$$

1. For $\lambda_1 = \lambda_2 = 1$, we recognize the Lie structure related to the Lie algebra $\mathfrak{g}_{3,3}$.
2. For $\lambda_1 = -\lambda_2 = 1$, we recognize the Lie structure related to the Lie algebra $\mathfrak{g}_{3,4}$.
3. For $\lambda_1 = 1$, $\lambda_2 = a$, we recognize the Lie structure related to the Lie algebra $\mathfrak{g}_{3,5}^a$.

Linear mappings and their eigenvectors giving three dimensional Lie algebras

F G	v w	Casimir	Name
$F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	x_1	$\mathfrak{g}_{3,1}$
$F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$x_1 e^{-\frac{x_2}{x_1}}$	$\mathfrak{g}_{3,2}$
$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\frac{x_2}{x_1}$	$\mathfrak{g}_{3,3}$
$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$x_1 x_2$	$\mathfrak{g}_{3,4}$
$F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$\frac{x_1}{x_2^a}$	$\mathfrak{g}^a_{3,5}$

$F = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$x_1^2 + x_2^2$	$\mathfrak{g}_{3,6}$
$F = \begin{pmatrix} a & -1 & 0 \\ 1 & a & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$	$(x_1^2 + x_2^2)e^{2a\mathrm{arctg}\frac{x_1}{x_2}}$	$\mathfrak{g}_{3,7}^a$
$F = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$x_1 x_3 + x_2^2$	$\mathfrak{g}_{3,8}$
$F = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $w = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$x_1^2 + x_2^2 + x_3^2$	$\mathfrak{g}_{3,9}$

General structure

For a Lie algebra $\mathfrak{g}$ with the basis $\{e_1, e_2, \ldots, e_N\}$, given by commutator relations $[e_i, e_j] = \sum_{k=1}^{N} c_{ij}^k e_k$, we can assign N-pairs $(F_1, e_N), \ldots, (F_{N-i+1}, e_i), \ldots, (F_N, e_1)$.

$$[e_i, e_N] = \sum_{k=1}^{N} c_{iN}^{k} e_k$$

$$F_1 = \left(\begin{array}{cccc|c} c_{1N}^{1} & c_{1N}^{2} & \cdots & c_{1N}^{N-1} & 0 \\ c_{2N}^{1} & c_{2N}^{2} & \cdots & c_{2N}^{N-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{N-1\ N}^{1} & c_{N-1\ N}^{2} & \cdots & c_{N-1\ N}^{N-1} & 0 \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array} \right),$$

The mapping F_1 corresponds as the vector e_n. In the above matrix, the structure constants c_{iN}^{N}, $i = 1, \ldots, N-1$ do not appear. They will be placed in the next mappings $F_2, \ldots, F_N$. To be precise, c_{iN}^{N} will appear in the mapping F_{N-i+1}.

$$F_1 = \left(\begin{array}{cccc|c} c^1_{1N} & c^2_{1N} & \dots & c^{N-1}_{1N} & 0 \\ c^1_{2N} & c^2_{2N} & \dots & c^{N-1}_{2N} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c^1_{N-1\ N} & c^2_{N-1\ N} & \dots & c^{N-1}_{N-1\ N} & 0 \\ \hline 0 & 0 & \dots & 0 & 0 \end{array}\right),$$

$$F_{N-i+1} = \left(\begin{array}{cccc|c|ccc} c^1_{1\ i} & c^2_{1\ i} & \dots & c^{i-1}_{1\ i} & 0 & c^{i+1}_{1\ i} & \dots & c^N_{1\ i} \\ c^1_{2\ i} & c^2_{2\ i} & \dots & c^{i-1}_{2\ i} & 0 & c^{i+1}_{2\ i} & \dots & c^N_{2\ i} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ c^1_{i-1\ i} & c^2_{i-1\ i} & \dots & c^{i-1}_{i-1\ i} & 0 & c^{i+1}_{i-1\ i} & \dots & c^N_{i-1\ i} \\ \hline 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & -c^{i+1}_{i\ i+1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & -c^N_{i\ N} \end{array}\right),$$

$$F_N = \left(\begin{array}{c|cccc} 0 & 0 & \dots & 0 \\ \hline 0 & -c_{1\,2}^{2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -c_{1\,N}^{N} \end{array}\right).$$

Theorem

Every Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ *is isomorphic to the corresponding Lie algebra* $(\mathbb{R}^n, [\cdot,\cdot]_{(F_1,v_1),\dots,(F_n,v_n)})$.

The isomorphism $(\mathfrak{g}, [\cdot,\cdot]) \cong (\mathbb{R}^n, [\cdot,\cdot]_{(F_1,v_1),\dots,(F_n,v_n)})$ is not canonical, we can assign the linear mappings and their eigenvectors differently.

Casimir functions

Using relations between the Lie algebra, the Lie–Poisson structure and the Nambu bracket, we show that the algebra invariants (Casimir functions) are solutions of an equation which has an interesting geometric significance.

Theorem

Casimir functions c_i, $i = 1, \ldots, k$, *for the Lie algebra* $(\mathbb{R}^n, [\psi, \phi]_{(F_1, v_1), \ldots, (F_n, v_n)})$ *satisfy the following equation*

$$\nabla c_i(\mathbf{x}) \wedge \star \sum_{j=1}^{n} (F_j(\mathbf{x}) \wedge v_j) = 0.$$

The Hodge star operator $\star : \bigwedge^2 V \longrightarrow \bigwedge^{n-2} V$

For a pair (F, v) giving a Lie algebra structure, we always have $N-2$ Casimir functions

Theorem

Let $(\mathbb{R}^N, [\cdot,\cdot]_{(F,e_N)})$ *be a Lie algebra, then Casimirs* $c_i, i = 1, 2, \ldots, N-2$, *of the algebra fulfill the following conditions*

$$\langle Fx|\nabla c_i(x)\rangle = 0,$$
$$\langle e_N|\nabla c_i(x)\rangle = 0$$

for all $x \in \mathbb{R}^N$.

A. Dobrogowska, M. Szajewska, *Eigenvalue problem versus Casimir functions for Lie algebras*, Anal. Math. Phys. **14** (2024), 1-24.

Casimir functions

Definition

A non-zero tensor $t \in \bigwedge^{N} V$ is s-partially decomposable if there exist $w_i, i = 1, 2, \ldots, s$, vectors and $N-s$-tensor $u \in \bigwedge^{N-s} V$ such that

$$t = w_1 \wedge w_2 \wedge \ldots \wedge w_s \wedge u.$$

Finally, the following theorem holds

Theorem

Let pairs (F_j, v_j), $j = 1, \ldots, N$, give any Lie algebra $\mathfrak{g}$. Functions c_i, $i = 1, \ldots, s$, are functionally independent Casimir functions for $\mathfrak{g}$ if and only if $\star \sum_{j=1}^{N} (F_j x \wedge v_j) \in \bigwedge^{N-2} \mathbb{R}^N$ is s-partially decomposable, i.e. if there exist $w_i \in \mathbb{R}^N, i = 1, 2, \ldots, s$, $u \in \bigwedge^{N-s-2} \mathbb{R}^N$ such that

$$\star \sum_{j=1}^{N} (F_j x \wedge v_j) = w_1 \wedge w_2 \wedge \ldots \wedge w_s \wedge u.$$

Furthermore, $\nabla c_i \sim w_i$.

If we have a single pair (F, e_N), then obviously the tensor $Fx \wedge e_N$ is decomposable, so consequently the tensor $\star(Fx \wedge e_N) \in \bigwedge^{N-2} \mathbb{R}^N$ is decomposable. Therefore, algebra with this pair must always have $N-2$ Casimir functions.

If there are $N-2$ smooth Casimir functions $c_1, \ldots, c_{N-2}$, this corresponds to the situation that the Poisson bracket arises from the Nambu bracket by fixing $N-2$ functions as Casimir functions. In this case, the formula has a form

$$\{f,g\}\Omega = u\; df \wedge dg \wedge dc_1 \wedge \ldots \wedge dc_{N-2}, \quad f, g \in C^\infty(\mathbb{R}^N),$$

where $\Omega = dx_1 \wedge \ldots \wedge dx_N$ is the standard volume element on $\mathbb{R}^N$, and u is some function on $\mathbb{R}^N$. The case, where there are less smooth Casimir functions, namely $c_1, \ldots, c_s$, $s < N-2$, then the Poisson bracket has a form

$$\{f,g\}\Omega = df \wedge dg \wedge dc_1 \wedge \ldots \wedge dc_s \wedge u.$$

In details studied in

P.A. Damianou, F. Petalidou, *Poisson Brackets with Prescribed Casimirs*, Canad. J. Math. **64** (5), (2012) 991–1018.

It is connected with $s+2$-linear Nambu bracket in dimension N, higher than $s+2$.

Chan C. Chandre, A. Horikoshi, *Classical Nambu brackets in higher dimensions*, J. Math. Phys. **64**, (2023) 052702.

Thank you for your attention