Geometric approach to the Moore-Penrose inverse and polar decomposition in operator ideals

E. Chiumiento [a] [b], P. Massey [a] [b]

[a]Centro de Matemática La Plata
UNLP

[b]Instituto Argentino de Matemática
"Alberto P. Calderón"

XLI Workshop on Geometric Methods in Physics
Białystok (2024)

Introduction

Let $\mathcal{H}$ be a separable complex infinite-dimensional Hilbert space.

A **symmetrically-normed ideal** is a two-sided ideal $\mathfrak{S} \subseteq \mathcal{B}(\mathcal{H})$ endowed with a norm $\|\cdot\|_{\mathfrak{S}}$ satisfying the following conditions:

- $\|ABC\|_{\mathfrak{S}} \leq \|A\|\|B\|_{\mathfrak{S}}\|C\|$, for all $A, C \in \mathcal{B}(\mathcal{H})$ and $B \in \mathfrak{S}$
- $\|B\|_{\mathfrak{S}} = \|B\|$, for every rank-one operator B
- $(\mathfrak{S}, \|\cdot\|_{\mathfrak{S}})$ is a Banach space

We consider proper s. n. ideals (i.e. $\mathfrak{S} \neq \{0\}$ and $\mathfrak{S} \neq \mathcal{B}(\mathcal{H})$).

Examples:

• $\mathcal{K} = \mathcal{K}(\mathcal{H})$ compact operators

• $\mathfrak{S}_p$ Schatten ideals ($p \geq 1$):

$$\mathfrak{S}_p = \left\{ T \in \mathcal{K} : \|T\|_p = \left(\sum_{n=1}^{\infty} s_n^p(T) \right)^{1/p} < \infty \right\}$$

• $\mathfrak{S}_\Phi$ ideal associated to a symmetric norming function Φ

Introduction

$\mathcal{CR}$ = Closed range operators

$\mathcal{CR}^+$ = Positive closed range operators

$\mathcal{PI}$ = Partial isometries

Moore-Penrose inverse: Given $B \in \mathcal{CR}$, the Moore-Penrose inverse $B^\dagger \in \mathcal{CR}$ is defined by

$$BB^\dagger B = B, \quad B^\dagger BB^\dagger = B^\dagger, \quad BB^\dagger = P_{R(B)}, \quad B^\dagger B = P_{N(B)^\perp}$$

Polar decomposition: $B = V_B|B|$, where $V_B \in \mathcal{PI}$, $|B| = (B^*B)^{1/2} \in \mathcal{CR}^+$

Fixed $A \in \mathcal{CR}$ and $\mathfrak{S}$ s. n. ideal, we consider perturbations

$$\mathcal{CR} \cap (A + \mathfrak{S}) = \{B \in \mathcal{CR} : B - A \in \mathfrak{S}\}$$

endowed with the metric

$$d_{\mathfrak{S}}(B_1, B_2) = \|B_1 - B_2\|_{\mathfrak{S}}, \quad B_1, B_2 \in \mathcal{CR} \cap (A + \mathfrak{S})$$

Introduction

We study the the following functions:

$$\mu : \mathcal{CR} \cap (A + \mathfrak{S}) \to \mathcal{CR},\ \mu(B) = B^\dagger$$ (**Moore-Penrose inverse**)

$$\alpha : \mathcal{CR} \cap (A + \mathfrak{S}) \to \mathcal{CR}^+,\ \alpha(B) = |B|$$ (**operator modulus**)

$$v : \mathcal{CR} \cap (A + \mathfrak{S}) \to \mathcal{PI},\ v(B) = V_B$$ (**polar factor**)

Some of the tools used in this work:

- Index of a pair of projections (or essential codimension)
- Infinite dim Lie groups and its homogeneous spaces associated to s. n. ideals
- Operator monotone functions

Continuity of the MP-inverse

Example: $B_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$, $B_n^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, $B = B^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Wedin's formula for matrices

$$A^\dagger - B^\dagger = -A^\dagger(A-B)B^\dagger + (A^*A)^\dagger(A^* - B^*)(I - BB^\dagger) + (I - A^\dagger A)(A^* - B^*)(BB^*)^\dagger$$

Remark: $\|A^\dagger - B^\dagger\| \leq (\|A^\dagger\|\|B^\dagger\| + \|A^\dagger\|^2\|B^\dagger\|^2)\|A-B\|$

Theorem (P. Wedin '73)

Let $\{B_n\}_{n\geq 1}$ be a sequence of matrices such that $\|B_n - B\| \to 0$. The following are equivalent:

i) $\|B_n^\dagger - B^\dagger\| \to 0$
ii) $sup_{n\geq 1}\|B_n^\dagger\| < \infty$
iii) $null(B_n) = null(B)$, *n large enough*

Continuity of the MP-inverse

Example: $B_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$, $B_n^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, $B = B^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Wedin's formula for matrices

$$A^\dagger - B^\dagger = -A^\dagger(A-B)B^\dagger + (A^*A)^\dagger(A^* - B^*)(I - BB^\dagger) + (I - A^\dagger A)(A^* - B^*)(BB^*)^\dagger$$

Remark: $\|A^\dagger - B^\dagger\| \leq (\|A^\dagger\|\|B^\dagger\| + \|A^\dagger\|^2\|B^\dagger\|^2)\|A - B\|$

Theorem (P. Wedin '73)

Let $\{B_n\}_{n\geq 1}$ be a sequence of matrices such that $\|B_n - B\| \to 0$. The following are equivalent:

i) $\|B_n^\dagger - B^\dagger\| \to 0$

ii) $sup_{n\geq 1}\|B_n^\dagger\| < \infty$

iii) $null(B_n) = null(B)$, *n large enough*

Continuity of the MP-inverse

Example: $B_n = \begin{pmatrix} 1 & 0 \\ 0 & 1/n \end{pmatrix}$, $B_n^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}$, $B = B^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Wedin's formula for matrices

$$A^\dagger - B^\dagger = -A^\dagger(A-B)B^\dagger + (A^*A)^\dagger(A^* - B^*)(I - BB^\dagger) + (I - A^\dagger A)(A^* - B^*)(BB^*)^\dagger$$

Remark: $\|A^\dagger - B^\dagger\| \leq (\|A^\dagger\|\|B^\dagger\| + \|A^\dagger\|^2\|B^\dagger\|^2)\|A - B\|$

Theorem (S. Izumino '83)

Let $\{B_n\}_{n\geq 1}$ in $\mathcal{CR}$ such that $\|B_n - B\| \to 0$, where $B \in \mathcal{CR}$. The following are equivalent:

i) $\|B_n^\dagger - B^\dagger\| \to 0$

ii) $sup_{n\geq 1}\|B_n^\dagger\| < \infty$

iii) $\|P_{N(B_n)} - P_{N(B)}\| < 1$, *n large enough*

Continuity of the MP-inverse

Given P, Q orthogonal projections such that $QP|_{R(P)} : R(P) \to R(Q)$ is a Fredholm operator, then (P, Q) is called a **Fredholm pair** and its **index** is

$$[P : Q] := ind(QP|_{R(P)} : R(P) \to R(Q)) \\ = dim(N(Q) \cap R(P)) - dim(R(Q) \cap N(P))$$

References: L. Brown, R. Douglas, P. Fillmore ’73 / S. Strătilă D. Voiculescu ‘78 / J. Avron, R. Seiler, B. Simon ’94/ W. Amrein, K. Sinha ’94/ E. Andruchow ‘14

Remark: $A, B \in \mathcal{CR}$, $A - B \in \mathfrak{S} \subset \mathcal{K} \Longrightarrow (P_{N(A)}, P_{N(B)})$ Fredholm pair

Theorem (E.C, P. Massey)

Let $\{B_n\}_{n\geq 1}$ in $\mathcal{CR}$ such that $\|B_n - B\|_{\mathfrak{S}} \to 0$, where $B \in \mathcal{CR}$. The following are equivalent:

i) $\|B_n^\dagger - B^\dagger\|_{\mathfrak{S}} \to 0$

ii) $sup_{n\geq 1}\|B_n^\dagger\| < \infty$

iii) $[P_{N(B_n)} : P_{N(B)}] = 0$, *n large enough*

Continuity sets of the MP-inverse

Let $Gl(\mathcal{H})$ be the group of invertible operators acting on $\mathcal{H}$, and let $\mathfrak{S}$ be a s. n. ideal, we consider the Banach-Lie group

$$\mathcal{G}\ell_{\mathfrak{S}} := \{G \in Gl(\mathcal{H}) : G - I \in \mathfrak{S}\},$$

where the topology is defined by $d_{\mathfrak{S}}(G_1, G_2) = \|G_1 - G_2\|_{\mathfrak{S}}$, $G_1, G_2 \in \mathcal{G}\ell_{\mathfrak{S}}$

Proposition (E.C., P. Massey)

Let $A \in \mathcal{CR}$ and let $\mathfrak{S}$ be a s. n. ideal. Then, we have

$$\mathcal{CR} \cap (A + \mathfrak{S}) = \bigcup_{k \in \mathbb{J}_A} \underbrace{\{B \in \mathcal{CR} : B - A \in \mathfrak{S},\ [P_{N(B)} : P_{N(A)}] = k\}}_{:= \mathcal{C}_k(A)}$$

where $\mathbb{J}_A = \{k \in \mathbb{Z} : -min\{dim\, N(A), dim\, R(A)^{\perp}\} \leq k \leq dim\, R(A)\}$. Furthermore, the following hold:

i) $(G, K) \cdot B = GBK^{-1}$, $G, K \in \mathcal{G}\ell_{\mathfrak{S}}$, $B \in \mathcal{C}_k(A)$, *transitive action on* $\mathcal{C}_k(A)$

ii) $\mu : \mathcal{C}_k(A) \to \mathcal{CR}$, $\mu(B) = B^{\dagger}$ *is locally Lipschitz*

Previous related works

Notation: $d_N(A,B) = \|A - B\| + \|P_{N(A)} - P_{N(B)}\|$

Theorem (G. Corach, A. Maestripieri, M. Mbekhta'09)

For $A \in \mathcal{CR}$, set $\mathcal{O}(A) = \{GAK^{-1} : G, K \in Gl(\mathcal{H})\}$. Then

$$\pi : Gl(\mathcal{H}) \times Gl(\mathcal{H}) \to (\mathcal{O}(A), d_N), \quad \pi((G,K)) = GAK^{-1}$$

admits continuous local cross sections.

Other works on MP-inverse and orbits of Banach-Lie groups associated to operator ideals:

• G.H. Golub, V. Pereyra '73: Directional derivatives of MP-inverse in sets of constant rank

• E. Andruchow, G. Corach, M. Mbekhta '05: Generalized inverses in C^*-algebras

• D. Beltiţă, T. Goliński, G. Jakimowicz, F. Pelletier '19: Understand MP-inverse as an inverse with certain pathologies

• D. Beltiţă, G. Larotonda '23: Unitary orbits of normal operators

MP-inverse as a real analytic map

Teorema (E.C, P. Massey)

Let $A \in \mathcal{CR}$ and let $\mathfrak{S}$ be a s. n. ideal. Then the following assertions hold:

i) $\pi : \mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{G}\ell_{\mathfrak{S}} \to (\mathcal{C}_k(A), d_{\mathfrak{S}})$, $\pi((G, K)) = GAK^{-1}$ *admits continuous local cross sections.*

ii) $\mathcal{C}_k(A)$ *is a real analytic homogeneous space of* $\mathcal{G}\ell_{\mathfrak{S}} \times \mathcal{G}\ell_{\mathfrak{S}}$, *and also a real analytic submanifold of* $A + \mathfrak{S}$

iii) *For each* $k \in \mathbb{J}_A$ *the map*

$$\mu : \mathcal{C}_k(A) \to \mathcal{C}_k(A^\dagger), \quad \mu(B) = B^\dagger,$$

is a real analytic map between manifolds.

1) Previous work (E.C. '10): 'Partial isometries $\Longleftrightarrow$ restricted Grassmannian'

Fixed $V \in \mathcal{PI}$, and the Banach-Lie group

$$\mathcal{U}_{\mathfrak{S}} = \{U \in \mathcal{U}(\mathcal{H}) : U - I \in \mathfrak{S}\}$$

We have

$$\mathcal{PI} \cap (V + \mathfrak{S}) = \bigcup_{k \in \mathbb{J}_V} \underbrace{\{W \in \mathcal{PI} : W - V \in \mathfrak{S},\ [P_{N(W)} : P_{N(V)}] = k\}}_{:= \mathcal{V}_k(V)}$$

The action $(U_1, U_2) \cdot W = U_1 W U_2^*$, $U_1, U_2 \in \mathcal{U}_{\mathfrak{S}}$, $W \in \mathcal{V}_k(V)$, is **transitive** on $\mathcal{V}_k(V)$

2) For $C \in \mathcal{CR}^+$, we have

$$\mathcal{CR}^+ \cap (C + \mathfrak{S}) = \bigcup_{k \in \mathbb{J}_C} \underbrace{\{B \in \mathcal{CR}^+ : B - C \in \mathfrak{S},\ [P_{N(B)} : P_{N(C)}] = k\}}_{:= \mathcal{P}_k(C)}$$

The action $G \cdot B = GBG^*$, $G \in \mathcal{G}\ell_{\mathfrak{S}}$, $B \in \mathcal{P}_k(C)$, is **transitive** on $\mathcal{P}_k(C)$

Operator modulus and polar factor

Problem: Let $A, B \geq 0$ such that $A - B \in \mathfrak{S}$. Do we have $A^{1/2} - B^{1/2} \in \mathfrak{S}$?

A function $f : [0, \infty) \to \mathbb{R}$ is **operator monotone** if $0 \leq C \leq D$ implies $f(C) \leq f(D)$.

Theorem (T. Ando, J.L. van Hemmen '80)

Let $C, D \geq 0$ such that $C^{1/2} + D^{1/2} \geq \mu > 0$, f operator monotone and $\mathfrak{S} = \mathfrak{S}_\Phi$

$\Longrightarrow \|f(D) - f(C)\|_{\mathfrak{S}} \leq K_{f,\mu} \|D - C\|_{\mathfrak{S}}$, *for some* $K_{f,\mu} > 0$

Theorem (E.C., P. Massey)

$f|_{\mathcal{P}_k(C)} : \mathcal{P}_k(C) \to f(C) + \mathfrak{S}$ *is real analytic for f operator monotone and* $C \in \mathcal{CR}^+$

Operator modulus and polar factor

Theorem (E.C., P. Massey)

Take $A \in \mathcal{CR}$ with polar decomposition $A = V_A|A|$, $k \in \mathbb{J}_A$ and $\mathfrak{S}$ s. n. ideal. Then the maps

$$\alpha : \mathcal{C}_k(A) \to \mathcal{P}_k(|A|), \;\; \alpha(B) = |B|,$$

and

$$v : \mathcal{C}_k(A) \to \mathcal{V}_k(V_A), \;\; v(B) = V_B,$$

are a real analytic fiber bundles

Thanks for your attention