Nijenhuis Geometry and Applications
Lecture 3

Conservation laws, symmetries and geodesically equivalent metrics

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XIII School on Geometry and Physics
Białystok, July 8 – July 12, 2024.

Plan

- Basic definitions (reminder)
- Symmetries and conservation laws
- Basic properties of symmetries and conservation laws of Nijenhuis operators
- Geodesically equivalent metrics and Nijenhuis operators
- Splitting theorem
- Local classification of geodesically equivalent metric and Levi-Civita theorem
- Singular points
- Integrability of some quasilinear systems
- Exercises

Basic definitions (for this lecture)

Definition (differential geometric)

A field of endomorphisms $L = (L^i_j)$ is called a *Nijenhuis operator*, if

$$\mathcal{N}_L(\xi, \eta) \stackrel{\text{def}}{=} L^2[\xi, \eta] - L[L\xi, \eta] - L[\xi, L\eta] + [L\xi, L\eta] = 0$$

for all vector fields ξ, η.

Definition (algebraic)

An operator $L : V \to V$, $\dim V = n$, is called *gl-regular*, if either of the following conditions holds:

- there is a vector ξ such that $\xi, L\xi, \dots, L^{n-1}\xi$ are linearly independent (such a vector is called *cyclic*);
- the operators $\mathrm{Id}, L, \dots, L^{n-1}$ form a basis of the centraliser of L;
- for each eigenvalue of L there is only one eigenvector;
- L can be reduced to the *first (or second) companion form*.

Theorem

Let f be a regular conservation law of L. Set $\mathrm{d}\, f_k = (L^*)^{k-1}\mathrm{d}\, f$ and consider $f_1, \dots, f_n$ as a (local) coordinate system. In these coordinates:

$$L_{\text{comp2}} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \dots & \sigma_2 & \sigma_1 \end{pmatrix}$$

Conversely, if $L = L_{\text{comp2}}$ in some coordinates $u_1, \dots, u_n$, then u_1 is a regular conservation law and $u_k = (L^*)^{k-1}\mathrm{d}\, u_1$.
Let $M = g_1 L^{n-1} + \dots + g_{n-1}L + g_n \mathrm{Id}$ be a regular symmetry of L and consider $g_1, \dots, g_n$ as a (local) coordinate system. In these coordinates:

$$L_{\text{comp1}} = \begin{pmatrix} \sigma_1 & 1 & & \\ \sigma_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix}$$

Conversely, if $L = L_{\text{comp1}}$ in some coordinates $u_1, \dots, u_n$, then $M = u_1 L^{n-1} + \dots + u_{n-1}L + u_n \mathrm{Id}$ is a regular symmetry.

Basic definitions (for this talk)

Let $A = (A^i_j)$ be an operator (not necessarily Nijenhuis).

Definition

A function f is a *conservation law* for A, if the form $A^* \mathrm{d} f$ is closed. (Today all the constructions are local so that this condition is equivalent to the existence of a function g such that $\mathrm{d} g = A^* \mathrm{d} f$.)

Definition

An operator $B = (B^i_j)$ is called a *strong symmetry* (resp. just *symmetry*) for the operator A, if

- $AB = BA$
- (i) *strong symmetry:*

$$\langle A, B\rangle(\xi, \eta) \stackrel{\mathrm{def}}{=} A[\xi, B\eta] + B[A\xi, \eta] - [A\xi, B\xi] - AB[\xi, \eta] = 0,$$

  (ii) *symmetry:*

$$\langle A, B\rangle(\xi, \xi) = A[\xi, B\xi] + B[A\xi, \xi] - [A\xi, B\xi] = 0.$$

Symmetries and conservation laws in dynamical systems

Instead of an operator A, consider a vector field ξ and the corresponding dynamical system

$$\frac{\mathrm{d}\, u}{\mathrm{d}\, t} = \xi(u) \qquad \text{in more detail:} \quad \frac{\mathrm{d}\, u^i}{dt} = \xi^i(u^1, \dots, u^n). \tag{1}$$

Definition

A *first integral* of ξ (or of the dynamical system (1)) is a function f such that $\xi(f) = \sum \xi^i \frac{\partial f}{\partial u^i} = 0$. In other words, $\mathcal{L}_\xi f = 0$.

Definition

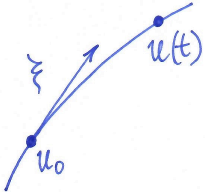
A *symmetry* (or *symmetry field*) of the system (1) is a vector field η such that $[\xi, \eta] = 0$. In other words, $\mathcal{L}_\xi \eta = 0$.

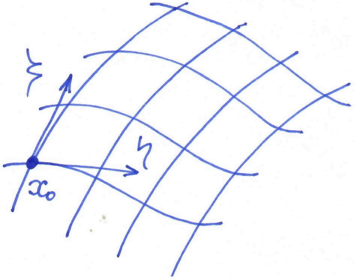
Properties.

- If η is a symmetry and f is an integral $\Rightarrow$ $f\eta$ is a symmetry and $\eta(f)$ is an integral.
- Let $\eta_1, \dots, \eta_k$ be linearly independent symmetries.
  The linear combination $\eta = \sum f_i \eta_i$ is a symmetry if and only if $f_1, \dots, f_k$ are first integrals.

Comments

The equation $\frac{\mathrm{d}\, u}{\mathrm{d}\, t} = \xi(u)$ describes an evolution of a point on the manifold.



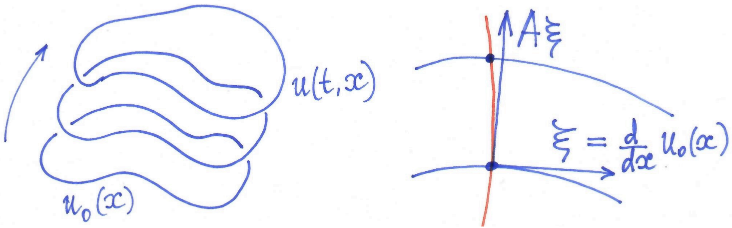


f is an integral $\Leftrightarrow$
$f(u(t)) = f(u_0)$

η is a symmetry $\Leftrightarrow$
for any point u_0
there is a surface $u(t, s)$
such that $\frac{\partial u}{\partial t} = \xi$, $\frac{\partial u}{\partial s} = \eta$

Comments

The equation $\dfrac{\partial u}{\partial t} = A(u)\dfrac{\partial u}{\partial x}$ describes an evolution of curves on the manifold.



- f is a conservation law $\Leftrightarrow \oint f(u(x,t))\mathrm{d}\,x = \oint f(u_0(x))\mathrm{d}\,x$, i.e. does not change under evolution.
- B is a symmetry $\Leftrightarrow$ for any initial curve $u_0(x)$ there is a “surface in the space of curves $u(t,s,x)$ such that

$$\frac{\partial u}{\partial t} = A(u)\frac{\partial u}{\partial x} \quad \text{и} \quad \frac{\partial u}{\partial s} = B(u)\frac{\partial u}{\partial x}\,.$$

Splitting theorem

Theorem

Assume that the characteristic polynomial $\chi_L(\lambda) = \det(\lambda \operatorname{Id} - L(\mathrm{p}))$ *of a Nijenhuis operator* L *at a point* p *splits into a product of two polynomials* $\chi_1(\lambda)$ *and* $\chi_2(\lambda)$ *with no common roots. Then there exists a coordinate system* $\underbrace{u^1, \dots, u^{m_1}}_{u}, \underbrace{v^1, \dots, v^{m_2}}_{v}$ *such that*

1. $L(u,v) = \begin{pmatrix} L_1(u) & 0 \\ 0 & L_2(v) \end{pmatrix}$, *where each* L_i *is a Nijenhuis operator and* $\chi_{L_i}(\lambda) = \chi_i(\lambda)$, $i = 1, 2$.
2. *Each conservation law* f *has the form* $f(u,v) = f_1(u) + f_2(v)$, *where* f_i *is a conservation law for* L_i, $i = 1, 2$.
3. *Each symmetry (resp. strong symmetry) has the form*

$$M(u,v) = \begin{pmatrix} M_1(u) & 0 \\ 0 & M_2(v) \end{pmatrix},$$

where M_i *is a symmetry (resp. strong symmetry) for* L_i, $i = 1, 2$.

Symmetries and conservation laws for a diagonal Nijenhuis operator

If $L = \mathrm{diag}(u_1, \dots u_n)$ or more generally

$$L = \mathrm{diag}(\lambda_1(u_1), \dots \lambda_n(u_n)),$$

where $\lambda_i(\cdot)$ are some functions (perhaps constant), satisfying $\lambda_i(u_i) \neq \lambda_j(u_j)$ almost everywhere, then the conservation laws and symmetries are very simple

$$f(u) = f_1(u_1) + f_2(u_2) + \dots + f_n(u_n)$$

and

$$M(u) = \begin{pmatrix} m_1(u_1) & & & \\ & m_2(u_2) & & \\ & & \ddots & \\ & & & m_n(u_n) \end{pmatrix}$$

$\mathfrak{gl}$-regular Nijenhuis operators

Theorem (Real analytic case)

Let L be a $\mathfrak{gl}$-regular Nijenhuis operator. Then there exist local coordinate systems $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ in which L reduces to the first and second companion forms:

$$L(u) = \begin{pmatrix} \sigma_1 & 1 & & \\ \vdots & 0 & \ddots & \\ \sigma_{n-1} & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \quad \text{and} \quad L(v) = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

where σ_i are the coefficients of the characteristic polynomial of L in the corresponding coordinate system.

Open problem. Does the statement of this theorem still hold in the C^∞-smooth case?

Symmetries of gl-regular Nijenhuis operators

If L is gl-regular, then every symmetry M can be uniquely written as

$$M = g_1 L^{n-1} + \dots + g_n \,\mathrm{Id}, \tag{2}$$

where g_i are some functions. We say that a symmetry M is *regular* at a point p, if the differentials $\mathrm{d}\, g_i$ are linearly independent at this point.

Theorem

Let L be a gl-regular Nijenhuis operator in a neighbourhood of p. Then

1. *Every symmetry M of L is strong.*
2. *For any two symmetries M_1 and M_2, their product $M_1 M_2$ is also a symmetry. In particular, the symmetries of a gl-regular operator L form an algebra w.r.t. pointwise matrix multiplication.*
3. *For any two symmetries M_1 and M_2, one has $\langle M_1, M_2 \rangle = 0$. In particular, every symmetry of L is a Nijenhuis operator.*
4. *Regular (local) symmetries of L are in one-to-one correspondence with the systems of first companion coordinates in the sense that the coefficients $g_1, \dots, g_n$ of expansion (2) are first companion coordinates for L if and only if M is a regular symmetry.*

Conservation laws of gl-regular Nijenhuis operators

Important property of Nijenhuis operators:

If $\mathrm{d}\,(L^*\mathrm{d}\,f) = 0$, then all the forms $(L^*)^k\mathrm{d}\,f$ are closed too.

This implies that (locally) f generates a *hierarchy of conservation laws* $f = f_1, f_2, f_3, \dots,$ where

$$(L^*)^i\mathrm{d}\,f_1 = f_{i+1} \text{ or, equivalently, } L^*\mathrm{d}\,f_i = \mathrm{d}\,f_{i+1}, \quad i = 1, \dots, n-1.$$

We say that a conservation law f (and the corresponding hierarchy) is *regular*, if $\mathrm{d}\,f_1, \dots, \mathrm{d}\,f_n$ are linearly independent.

Theorem

Let L be a gl-regular Nijenhuis operator in a neighbourhood of p. Then

1. *Every conservation law $\mathrm{d}\,f$ of L is a conservation law for all of its symmetries, that is, $\mathrm{d}\,(M^*\mathrm{d}\,f) = 0$ for any symmetry M.*
2. *Regular (local) hierarchies of conservation laws of L are in one-to-one correspondence with systems of second companion coordinates, in the sense that $f_1, f_2, f_3, \dots$ is a regular hierarchy if and only if $f_1, \dots, f_n$ are second companion coordinates for L.*

Explicit parametrisation for symmetries and conservation laws and relationship between them

Theorem

Let L be a real analytic Nijenhuis operator, gl-regular at a point p. Then

1. *There exists a regular symmetry U centred at p, and a regular conservation law f.*
2. *For any collection of functions v_i analytic in a neighbourhood of $0 \in \mathbb{R}$, and any regular symmetry U centred at p, the operator*

$$M = v_1(U)L^{n-1} + \cdots + v_{n-1}(U)L + v_n(U) \tag{3}$$

   is a symmetry. Moreover, every symmetry of L can be written in this form with an appropriate choice of functions v_i.
3. *Given a regular conservation law f, for any conservation law h there exists a symmetry M such that $\mathrm{d}\, h = M^* \mathrm{d}\, f$.*

Jordan block in dimension 3 (example)

Useful formula from Linear Algebra:

$$f(L_{nc}) = f\left(\begin{pmatrix} u_3 & u_2 & u_1 \\ 0 & u_3 & u_2 \\ 0 & 0 & u_3 \end{pmatrix}\right) = \begin{pmatrix} f(u_3) & f'(u_3)u_2 & f'(u_3)u_1 + \frac{f''(u_3)}{2}u_2^2 \\ 0 & f(u_3) & f'(u_3)u_2 \\ 0 & 0 & f(u_3) \end{pmatrix} =$$

$$\begin{pmatrix} f & g & h \\ 0 & f & g \\ 0 & 0 & f \end{pmatrix}, \qquad \text{where} \qquad \begin{aligned} f &= f(u_3) \\ g &= g(u_2, u_3) = f'(u_3)u_2 \\ h &= h(u_1, u_2, u_3) = f'(u_3)u_1 + \frac{f''(u_3)}{2}u_2^2 \end{aligned}$$

Symmetry of general type:

$$M = f_1(L_{nc})L_c^2 + f_2(L_{nc})L_c + f_3(L_{nc}) = \begin{pmatrix} f_3 & g_3 + f_2 & h_3 + g_2 + f_1 \\ 0 & f_3 & g_3 + f_2 \\ 0 & 0 & f_3 \end{pmatrix}$$

Conservation law of general type:

$$f(u_1, u_2, u_3) = h_3 + g_2 + f_1 = f_3'(u_3)u_1 + \tfrac{1}{2}f_3''(u_3)u_2^2 + f_2'(u_3)u_2 + f_1(u_3)$$

Summary

Symmetries and conservation laws of a $\mathfrak{gl}$-regular Nijenhuis operator L possess several remarkable properties:

- P1. Each symmetry of L is strong.
- P3. Each symmetry of L is Nijenhuis.
- P2. If M_1 and M_2 are symmetries of L, then their product $M_1 M_2$ is a symmetry also.
- P4. Symmetries M_1 and M_2 commute is the algebraic sense, i.e., $M_1 M_2 = M_2 M_1$, and are symmetries of each other.
- P5. Every conservation law f of the operator L is a conservation law for each of its symmetry M, that is, $\mathrm{d}\,(M^* \mathrm{d}\, f) = 0$.
- P6. Let f be a regular conservation law of L. Then any other conservation law h can be obtained from $\mathrm{d}\, g = M^* \mathrm{d}\, f$, where M is a suitable symmetry of L.

gl-regularity is essential: Example

Consider the constant operator $L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in $\mathbb{R}^3(x, y, z)$, which consists of two nilpotent Jordan blocks of size 2 and 1.

The symmetries of L have the following form

$$M = \begin{pmatrix} f & xf_y + g & xf_z + a \\ 0 & f & 0 \\ 0 & b & c \end{pmatrix},$$

where the functions f, g, a, b, c depend on y and z only.
Strong symmetries have a similar form with the additional condition that $f = f(y)$ (i.e., f does not depend on z).

The conservation laws are $xu(y) + v(y, z)$.

None of the properties P1 – P5 are met.

Applications to geodesically equivalent metrics

Definition

Two (pseudo)-Riemannian metrics g and $\bar{g}$ are called *geodesically equivalent* if they share the same geodesics viewed as unparameterized curves.

A manifold endowed with a pair of such metrics carries a natural Nijenhuis structure

$$L = \left| \frac{\det \bar{g}}{\det g} \right|^{\frac{1}{n+1}} \bar{g}^{-1} g.$$

In terms of L, the geodesic equivalence condition is given by the PDE equation

$$\nabla_\eta L = \frac{1}{2}\big(\eta \otimes \mathrm{d}\, \mathrm{tr}\, L + (\eta \otimes \mathrm{d}\, \mathrm{tr}\, L)^*\big), \tag{4}$$

where η is an arbitrary vector field.

Definition

If (4) holds, then the metric g and Nijenhuis operator L are said to be *geodesically compatible*.

Why is L a Nijenhuis operator?

Proposition

Let L be g-symmetric and satisfy $\nabla_\eta L = \frac{1}{2}\big(\eta \otimes \mathrm{d}\,\mathrm{tr}\, L + (\eta \otimes \mathrm{d}\,\mathrm{tr}\, L)^*\big)$.
Then L is Nijenhuis.

Proof.

The Nijenhuis torsion $N_L(\xi,\eta) = L^2[\xi,\eta] - L[L\xi,\eta] - L[\xi, L\eta] + [L\xi, L\eta]$ can be naturally expressed in terms of any symmetric connection ∇. Namely,

$$N_L(\xi,\eta) = (L\nabla_\eta L - \nabla_{L\eta} L)\xi - (L\nabla_\xi L - \nabla_{L\xi} L)\eta.$$

It remains to substitute (we denote $\theta = \mathrm{d}\,\mathrm{tr}\, L$ and $\theta^\sharp = g^{-1}\theta$)

$$\begin{aligned}
2N_L(\xi,\eta) &= \Big(L(\eta\otimes\theta + (\eta\otimes\theta)^*) - (L\eta\otimes\theta + (L\eta\otimes\theta)^*)\Big)\xi \\
&- \Big(L(\xi\otimes\theta + (\xi\otimes\theta)^*) - (L\xi\otimes\theta + (L\xi\otimes\theta)^*)\Big)\eta \\
&= L(\eta\otimes\theta)^*\xi - L(\xi\otimes\theta)^*\eta - (L\eta\otimes\theta)^*)\xi + (L\xi\otimes\theta)^*)\eta \\
&= g(\eta,\xi)\, L\theta^\sharp - g(\xi,\eta)\, L\theta^\sharp - g(L\eta,\xi)\,\eta^\sharp + g(L\xi,\eta)\,\eta^\sharp = 0.
\end{aligned}$$

as required. $\square$

Splitting-gluing theorem

Theorem

Let h_i (metric) and L_i (Nijenhuis operator) be geodesically compatible, $i = 1, 2$, and χ_i be the characteristic polynomial of L_i.
Then

$$L = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} h_1(x)\chi_2\big(L_1(x)\big) & 0 \\ 0 & h_2(y)\chi_1\big(L_2(y)\big) \end{pmatrix}$$

Notice that $g_1(x, y) = h_1(x)\chi_2\big(L_1(x)\big)$ and $g_2(x, y) = h_2(y)\chi_1\big(L_2(y)\big)$ so that the variables x and y are mixed in g_1 and g_2.

Conversely, assume that L and g are geodesically compatible and L splits into direct product $L(x, y) = L_1(x) \oplus L_2(y)$. Then g has the above form for some h_1 and h_2 geodesically compatible with L_1 and L_2 respectively.

Dini theorem

In dimension 1: $h = f(x)\mathrm{d}x^2$, $L = x\,\mathrm{d}x \otimes \frac{\partial}{\partial x}$ are always compatible.
Taking two copies of this trivial exampe

$$h_1 = f(x)\mathrm{d}x^2, L_1 = x\,\mathrm{d}x \otimes \tfrac{\partial}{\partial x} \quad \text{and} \quad h_2 = g(y)\mathrm{d}y^2, L_2 = y\,\mathrm{d}y \otimes \tfrac{\partial}{\partial y}$$

or in matrix form

$$h_1 = \begin{pmatrix} f(x) \end{pmatrix},\ L_1 = \begin{pmatrix} x \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} g(y) \end{pmatrix},\ L_2 = \begin{pmatrix} y \end{pmatrix}$$

Applying the gluing procedure gives

$$L = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad g = \begin{pmatrix} f(x)(y-x) & 0 \\ 0 & g(y)(x-y) \end{pmatrix}$$

or, in more standard form,

$$g = (y-x)\Big(f(x)\,\mathrm{d}x^2 - g(y)\,\mathrm{d}y^2\Big)$$

and using $\bar{g} = \frac{1}{\det L} g L^{-1}$:

$$\bar{g} = \left(\frac{1}{x} - \frac{1}{y}\right)\left(\frac{f(x)}{x}\,\mathrm{d}x^2 - \frac{g(y)}{y}\,\mathrm{d}y^2\right)$$

Levi-Civita theorem

Theorem

Let g and $\bar{g}$ be geodesically equivalent and such that the operator $\bar{g}g^{-1}$ has different non-constant eigenvalues. Then

$$g = \sum_i \frac{\prod_{\alpha \neq i}(x_i - x_\alpha)}{f_i(x_i)} \mathrm{d}x_i^2$$

and

$$\bar{g} = \frac{1}{\det L} g L^{-1}, \quad \textit{with } L = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$$

Integrability: from L to the integrals of the geodesic flow

Simple observation. If L and g are geodesically compatible, then $L - t \cdot \mathrm{Id}$ and g are geodesically compatible also. A metric g may admit many geodesically compatible operators. They form a finite-dimensional vector space.

Theorem (Benenti, Matveev, Topalov, Tabachnikov, ...)

Let L be geodesically compatible with g. Then the geodesic flow of g admits a (quadratic in momenta) first integral of the form

$$F(x,p) = \det L \cdot g^{-1}\big((L^*)^{-1}p, p\big)$$

More generally, any function of the form (we just replace L with $L - t\,\mathrm{Id}$)

$$\det(L - t\,\mathrm{Id}) \cdot g^{-1}\big((L^* - t\,\mathrm{Id})^{-1}p, p\big) = \sum_{k=1}^{n} F_k(x,p)t^{n-k}$$

is a first integral too.
The functions $F_1, \dots, F_n$ Poisson commute and are independent if L is gl-regular. In particular, if L is gl-regular, then the geodesic flow of L is completely integrable.

Proof

We identify T^*M with TM by setting $p_i = g_{ij}\dot{x}$ and verify that $F(x,p) = F(x,\xi) = g(\det L \cdot L^{-1}\xi, \xi)$ is a first integral of the geodesic flow of g by straightforward computation:

$$\frac{\mathrm{d}}{\mathrm{d}\, t} g(\det L \cdot L^{-1}\xi, \xi) = \nabla_\xi g(\det L \cdot L^{-1}\xi, \xi) = g(\nabla_\xi(\det L \cdot L^{-1})\xi), \xi)$$

Next,

$$\begin{aligned}\nabla_\xi\left(\det L \cdot L^{-1}\right) &= \left(\mathcal{L}_\xi \det L\right) \cdot L^{-1} - \det L \cdot \nabla_\xi L^{-1} \\ &= \det L \left((\mathcal{L}_\xi \ln\det L) \cdot L^{-1} - L^{-1}(\nabla_\xi L) L^{-1}\right).\end{aligned}$$

Finally, we use the following general property of Nijenhuis operators $\mathcal{L}_u \ln\det L = \mathcal{L}_{L^{-1}u} \operatorname{tr} L$ and the geodesic compatibility condition (4) in the form $g((\nabla_\xi L)\eta, \eta) = \mathcal{L}_\eta \operatorname{tr} L\, g(\xi, \eta)$ to get

$$g\left((\nabla_\xi \det L \cdot L^{-1})\xi, \xi\right) =$$
$$\det L \cdot \left(\mathcal{L}_{L^{-1}\xi} \operatorname{tr} L \cdot g\left(L^{-1}\xi, \xi\right) - g\left((\nabla_\xi L)L^{-1}\xi, L^{-1}\xi\right)\right) =$$
$$= \det L \cdot \left(\mathcal{L}_\eta \operatorname{tr} L \cdot g\left(\xi, \eta\right) - g\left((\nabla_\xi L)\eta, \eta\right)\right) = 0,$$

as required.

Proof (continued....)

To verify Poisson commutativity, we may use the following fact

Lemma

Let A and B be Killing $(1,1)$-tensors for a metric g, i.e., the quadratic functions $F_A = g^{-1}(A^* p, p)$ and $F_B = g^{-1}(B^* p, p)$ are first integrals of the geodesic flow of g. Assume that A and B are symmetries of each other, then F_A and F_B Poisson commute.

In our situation, it can be checked that the operators

$$A_f = \det(L - t\,\mathrm{Id})(L - t\,\mathrm{Id})^{-1}$$

are all symmetries of each other. Hence, the Poisson commutativity follows.

Why not to add a potential?

Theorem

Let L and g be geodesically compatible. Assume that L is gl-regular and consider an arbitrary symmetry M of L:

$$M = U_1 L^{n-1} + U_2 L^{n-2} + \cdots + U_n \mathrm{Id}, \tag{5}$$

where the coefficients $U_1, \ldots, U_N$ are uniquely defined smooth functions. Then the geodesic flow g with the potential U_1, i.e., the Hamiltonian system with the Hamiltonian

$$H(x, p) = \tfrac{1}{2} g^{-1}(p, p) + U_1(x) = F_1(x, p) + U_1(x) \tag{6}$$

is completely integrable by means of the following commuting first integrals (quadratic in momenta)

$$\widetilde{F}_k(p, x) = F_k(p, x) + U_k(x), \quad k = 1, \ldots, n. \tag{7}$$

Some useful formulas

We deal with the following objects:

- metric $g = (g_{ij})$
- Hamiltonian of the geodesic flow $H = \frac{1}{2} g^{-1}(p, p) = \frac{1}{2} \sum g^{ij}(x) p_i p_j$
- Killing 2-tensor $K = (K_{ij})$: $\nabla_i K_{jk} + \nabla_j K_{ki} + \nabla_k K_{ij} = 0$
- quadratic first integral $F = f^{ij}(x) p_i p_j$: $\quad \{H, F\} = 0$
- geodesically equivalent metric $\bar{g} = (\bar{g}_{ij})$
- Nijenhuis operator L geodesically compatible with g:

We may think of these objects as $n \times n$-matrices. They are related to each other in a certain way...

- $F = g^{-1} K g^{-1}$
- $K = \det L \cdot g L^{-1}$
- $\bar{g} = \frac{1}{\det L} \cdot g L^{-1}$
- $L = \left(\frac{\det \bar{g}}{\deg g} \right)^{\frac{1}{n+1}} g \bar{g}^{-1}$
- $L = (\det g \det F)^{\frac{1}{n-1}} g^{-1} F^{-1}$

Even simpler formulas for the standard Euclidean metric in dimension 2

Let $g = \mathrm{d}\,x^2 + \mathrm{d}\,y^2$ be the standard Euclidean metric.
In matrix form $g = \mathrm{Id}$.

We consider

- first integral $F = ap_x^2 + 2bp_xp_y + cp_y^2$
- Killing 2-tensor $K = k_{11}\,\mathrm{d}x^2 + 2k_{12}\,\mathrm{d}x\mathrm{d}y + k_{22}\,\mathrm{d}y^2$
- geodesically equivalent $\bar{g} = \bar{g}_{11}\,\mathrm{d}x^2 + 2\bar{g}_{12}\,\mathrm{d}x\mathrm{d}y + \bar{g}_{22}\,\mathrm{d}y^2$
- geodesically compatible $L = \begin{pmatrix} l_1^1 & l_2^1 \\ l_1^2 & l_2^2 \end{pmatrix}$

Then, thinking of F, K, $\bar{g}$, L as (symmetric) 2×2 matrices, we get

- $\det F = \det K = \det L = (\det \bar{g})^{-\frac{1}{3}}$
- $K = F = \frac{1}{(\det \bar{g})^{2/3}}\,\bar{g} = \det L \cdot L^{-1} = \operatorname{adj} L$
- $\bar{g} = \frac{1}{\det L} L^{-1} = \frac{1}{(\det F)^2} F$
- $L = (\det \bar{g})^{\frac{1}{3}}\,\bar{g}^{-1} = \det F \cdot F^{-1} = \operatorname{adj} F$

Simple example: verification

Example

Let $g = \mathrm{d}\,x^2 + \mathrm{d}\,y^2$ be the standard Euclidean metric. Then $F = ap_x^2 + 2bp_xp_y + cp_y^2$ is a first integral of the geodesic flow of g if and only if $L = \operatorname{adj} F = \begin{pmatrix} c & -b \\ -b & a \end{pmatrix}$ is geodesically compatible with g,

Verification:

$$\begin{aligned} \{H, F\} &= \{\tfrac{1}{2}(p_x^2 + p_2^2), ap_x^2 + 2bp_xp_y + cp_y^2\} \\ &= a_xp_x^2 + (2b_x + a_y)p_x^2p_y + (2b_y + c_x)p_xp_y^2 + c_yp_y^3 = 0 \end{aligned}$$

if and only if $a_x = 0, \quad 2b_x + a_y = 0, \quad 2b_y + c_x = 0, \quad c_y = 0.$

On the other hand, $\nabla_\eta L = \frac{1}{2}\left(\eta \otimes \mathrm{d}\ \mathrm{tr}\, L + (\eta \otimes \mathrm{d}\ \mathrm{tr}\, L)^\top\right)$ with arbitrary $\eta = (\eta_1, \eta_2)$ gives

$$\begin{pmatrix} \eta_1 c_x + \eta_2 c_y & -2(\eta_1 b_x + \eta_2 b_y) \\ * & \eta_1 a_x + \eta_2 a_y \end{pmatrix} = \begin{pmatrix} \eta_1(a_x + c_x) & \frac{1}{2}(\eta_2(a_x + c_x) + \eta_1(a_y + c_y)) \\ * & \eta_2(a_y + c_y) \end{pmatrix}$$

leading to the same conditions $a_x = 0$, $2b_x + a_y = 0$, $2b_y + c_x = 0$, $c_y = 0$.

Further useful formulas in dimension 2

Let $g = \mathrm{d}\,x^2 + \mathrm{d}\,y^2$ be the standard Euclidean metric. In matrix form $g = \mathrm{Id}$.

A quadratic Hamiltonian of the geodesic flow of g is an arbitrary quadratic form with constant coefficients of the linear integrals (Killing vector fields)

$$p_x, \quad p_y, \quad xp_y - yp_x.$$

(Recall that in this setting, the operators L must be invertible, and this property is easy to achieve by $L \mapsto L + c\,\mathrm{Id}$.)

Case 1. $$F = 2p_y(xp_y - yp_x) \quad \text{or, in matrix form, } F = \begin{pmatrix} 0 & -y \\ -y & 2x \end{pmatrix}$$

$$L = \det F \cdot F^{-1} = \operatorname{adj} F = \begin{pmatrix} 2x & y \\ y & 0 \end{pmatrix}$$

Case 2. $$F = (xp_y - yp_x)^2 \quad \text{or, in matrix form, } F = \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}$$

$$L = \det F \cdot F^{-1} = \operatorname{adj} F = \begin{pmatrix} x^2 & yx \\ yx & y^2 \end{pmatrix}$$

Geodesically equivalent metrics for $\mathrm{d}x^2 + \mathrm{d}y^2$

Proposition

Let $F = ap_x^2 + 2b\, p_x p_y + c\, p_y^2$ be a first integral for the geodesic flow of the standard Euclidean metric $\mathrm{d}x^2 + \mathrm{d}y^2$. Then the metric

$$\mathrm{d}\, s^2 = \frac{1}{(ac - b^2)^2}(a\,\mathrm{d}x^2 + 2b\,\mathrm{d}x\mathrm{d}y + c\,\mathrm{d}y^2)$$

is geodesically equivalent to $\mathrm{d}x^2 + \mathrm{d}y^2$.

Two specific examples related to Case 1 and Case 2.

Case 1. $\frac{1}{(c^2+2cx-y^2)^2}\Big(c\,\mathrm{d}x^2 - 2y\,\mathrm{d}x\mathrm{d}y + (2x+c)\,\mathrm{d}y^2\Big)$

Case 2. $\frac{1}{\big(c^2+c(x^2+y^2)\big)^2}\Big((c+y^2)\,\mathrm{d}x^2 - 2xy\,\mathrm{d}x\mathrm{d}y + (c+x^2)\,\mathrm{d}y^2\Big)$

Singularities in the context of geodesically equivalent metrics

Singular points are those at which the algebraic type of L changes, e.g., the eigenvalues of L collide.

Open problem. What kind of singular points can appear in the context of geodesically equivalent metrics?

Example. $\begin{pmatrix} x & y \\ y & 0 \end{pmatrix}$ is allowed, $\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$ is not.

If L is a gl-regular operator, then its eigenvalues can still collide without violating the gl-regularity condition. In the Nijenhuis geometry, scenarios of such collisions can be very different. However, regardless of any particular scenario, we have the following general local result.

Theorem

Let L be a gl-regular real analytic Nijenhuis operator. Then (locally) there exists a pseudo-Riemannian metric g geodesically compatible with L. Moreover, such a metric g can be defined explicitly in terms of the second companion form of L.

Magic formula (Konyaev)

Fix second companion coordinates $u^1, \dots, u^n$ of L so that

$$L = L_{\text{comp2}} = \begin{pmatrix} 0 & 1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \dots & 0 & 1 \\ \sigma_n & \sigma_{n-1} & \dots & \sigma_1 \end{pmatrix},$$

Let $p_1, \dots, p_n, u^1, \dots, u^n$ be the corresponding canonical coordinates on the cotangent bundle and consider the following algebraic identity

$$h_1 L^{n-1} + \dots + h_n \,\mathrm{Id} = \Big(p_n L^{n-1} + \dots + p_1 \,\mathrm{Id}\Big)^2. \tag{8}$$

Since L is gl-regular, the functions $h_1, \dots, h_n$ are uniquely defined. They are quadratic in $p_1, \dots, p_n$ and their coefficients are polynomials in σ_i's.

Proposition

The function $h_1(u,p) = \sum h_1^{\alpha\beta}(u) p_\alpha p_\beta$ defines a non-degenerate (contravariant) metric which is geodesically compatible with L.

How to describe all geodesically compatible partners for L?

Let L be an admissible Nijenhuis operator (in the context of geodesic equivalence), i.e. there is at least one (pseudo)-Riemannian metric g geodesically compatible with L.

Open problem. Describe all geodesically compatible partners for L.

Theorem

Let L and g be geodesically compatible. Assume that M is g-symmetric and is a strong symmetry of L, then L and $gM := (g_{is}M^s_j)$ are geodesically compatible.
Moreover, if L is gl-regular, then every metric $\tilde{g}$ geodesically compatible with L is of the form $\tilde{g} = gM$, where M is a (strong) symmetry of L.

Quasilinear systems related to Nijenhuis operators

For a given Nijenhuis operator L, we define the operator fields A_i by the following recursion relations

$$A_0 = \mathrm{Id}, \quad A_{i+1} = LA_i - \sigma_i \,\mathrm{Id}, \quad i = 0, \dots, n-1, \tag{9}$$

where functions σ_i are coefficients of the characteristic polynomial of L numerated as below:

$$\chi_L(\lambda) = \det(\lambda \,\mathrm{Id} - L) = \lambda^n - \sigma_1 \lambda^{n-1} - \dots - \sigma_n. \tag{10}$$

Equivalently, the operators A_i can be defined from the matrix relation

$$\det(\lambda \,\mathrm{Id} - L) \cdot (\lambda \,\mathrm{Id} - L)^{-1} = \lambda^{n-1} A_0 + \lambda^{n-2} A_1 + \dots + \lambda A_{n-2} + A_{n-1}.$$

Consider the following system of quasilinear PDEs defined by these operators

$$\begin{aligned} u_{t_1} &= A_1 \, u_x, \\ &\dots \\ u_{t_{n-1}} &= A_{n-1} \, u_x, \end{aligned} \tag{11}$$

with $u^i = u^i(x, t_1, \dots, t_{n-1})$ being unknown functions in n variables and $u = (u^1, \dots, u^n)^\top$.

Finite dimensional reductions

Informally, a finite-dimensional reduction of an integrable PDE system is a subsystem of it, which is finite-dimensional and still integrable.
It appears that such a reduction of (11) can be naturally obtained by fixing a metric g geodesically compatible with L.

Theorem

Consider any metric g geodesically compatible with L and take any geodesic $\gamma(x)$ of this metric. Let $u(x, t_1, ..., t_{n-1})$ be the solution of (11) with the initial condition $u(x, 0, ..., 0) = \gamma(x)$. Then for any (sufficiently small) $t_1, ..., t_{n-1}$, the curve $x \mapsto u(x, t_1, ..., t_{n-1})$ is a geodesic of g.

In other words, the evolutionary system corresponding to any of the equations from (11) sends geodesics of g to geodesics.

Explanation. The integrals of the geodesic flow of g are closely related to the operators A_i (V. Matveev). Namely, if g is geodesically compatible with L, then its geodesic flow (as a Hamiltonian system on T^*M) admits n commuting first integrals $F_0, \dots, F_{n-1}$ of the form

$$F_i(u, p) = \tfrac{1}{2}\, g^{-1}(A_i^* p, p). \tag{12}$$

Finite dimensional reductions (continued...)

Let us consider the space $\mathfrak{G}$ of all g-geodesics (viewed as parameterised curves). Then system (11) defines a local action of $\mathbb{R}^n$ on $\mathfrak{G}$:

$$\Psi^{t_0,t_1,\dots,t_{n-1}} : \mathfrak{G} \to \mathfrak{G}, \qquad (t_0, t_1, \dots, t_{n-1}) \in \mathbb{R}^n.$$

More precisely, if $\gamma = \gamma(x) \in \mathfrak{G}$ is a g-geodesic, then we set $\Psi^{t_0,t_1,\dots,t_{n-1}}(\gamma)$ to be the geodesic $\tilde{\gamma}(x) = u(x + t_0, t_1, ..., t_{n-1})$, where $u(x, t_1, ..., t_{n-1})$ is the solution of (11) with the initial condition $u(x, 0, ..., 0) = \gamma(x)$.

Theorem

The action Ψ is conjugate to the Hamiltonian action of $\mathbb{R}^n$ on $T^*\mathrm{M}$ generated by the flows of the integrals $F_0, \dots, F_{n-1}$ defined by (12). *The conjugacy is given by $\gamma \in \mathfrak{G} \mapsto (\gamma(0), g_{ij}\dot{\gamma}^i(0)) \in T^*\mathrm{M}$.*

Remark. Let L be a gl-regular real analytic Nijenhuis operator, then for every curve γ with a cyclic velocity vector there exists a metric g geodesically compatible with L such that γ is a g-geodesic. Thus, the above finite-dimensional reductions of (11) 'cover' almost all (local) solutions of the Cauchy problem.

Symmetries and conservation laws of (11)

Theorem

If L is gl-regular, then

1. *For any hierarchy of conservation laws $f_1, \ldots, f_n$ of L, the operator*

$$B = f_1 A_n + \cdots + f_n A_1$$

   is a common symmetry for A_i. Moreover, every common symmetry of A_i's can be written in this way.

2. *For any symmetry $M = g_1 L^{n-1} + \cdots + g_n \,\mathrm{Id}$ of L, the first function g_1 is a common conservation law of A_i.*
   Moreover, every common conservation law of A_i's can be obtained in this way.

Exercises

- Prove the equivalence of all the conditions from Definition of gl-regular operators.
- Let $L = \begin{pmatrix} f(x,y) & 1 \\ g(x,y) & 0 \end{pmatrix}$. Find necessary and sufficient conditions for L to be a Nijenhuis operator in terms of f and g.
- Let $L = \begin{pmatrix} 0 & 1 \\ g(x,y) & f(x,y) \end{pmatrix}$. Find necessary and sufficient conditions for L to be a Nijenhuis operator in terms of f and g.
- Give an example of a (non-Nijenhuis) operator A that cannot be reduced to a companion form.
- The proof of the Splitting Theorem for conservation laws and symmetries uses the following algebraic fact. Let A and B be square matrices of sizes $n \times n$ and $k \times k$. Prove that the matrix equation $AX = XB$ (for an unknown $n \times k$ matrix X) has only trivial solution $X = 0$ if and only if A and B have no common eigenvalues.
- Prove the 'recursion property' of Nijenhuis operators: let f_0 be a conservation law of a Nijenhuis operator L, i.e. the form $\mathrm{d}\,(L^* \mathrm{d}\, f_0)$ is closed. Then there locally exist functions $f_1, f_2, f_3, \dots$ such that $\mathrm{d}\, f_k = L^* \mathrm{d}\, f_{k-1}$.

Exercises

- Let L be a Nijenhuis operator and $f = \operatorname{tr} L$. Find g such that $\mathrm{d}\, g = L^* \mathrm{d}\, f$ and construct the hierarchy of conservation laws generated by $f = \operatorname{tr} L$.
- Let L be a differentially non-degenerate Nijenhuis operator in dimension n=2. Construct a second companion coordinate system in terms of the coefficient of the characteristic polynomial of L. The same question for $n = 3$ and arbitrary n.
- Let $L = \begin{pmatrix} 0 & x \\ 0 & y \end{pmatrix}$. Check that L is Nijenhuis and describe all conservation laws and symmetries of L. Prove that each symmetry of L is strong.