Nijenhuis Geometry and Applications
Lecture 2
Linearisation and left-symmetric algebras

Alexey Bolsinov
Loughborough University, UK

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Plan

- Linearisation at singular points
- Left-symmetric algebras
- Comparison with Poisson Geometry
- Some properties of left-symmetric algebras
- Linearisation and non-degeneracy problems
- Some classification results
- Non-degeneracy of the diagonal algebra
- Interesting open problem
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Linearisation at singular points

Let L be a Nijenhuis operator on a manifold M.

Definition

A singular point $p \in M$ is said to be of *scalar type*, if $L(p) = \lambda \cdot \mathrm{Id}$.

Assume (w.l.o.g.) that $\lambda = 0$. Then locally:

$$L(x) = 0 + L_1(x) + L_2(x) + L_3(x) + \dots$$

where the entries of $L_k(x)$ are homogeneous polynomials in $x^1, \dots, x^n$ of degree k.

Proposition (Definition)

The linear part $L_{\mathrm{lin}} = L_1 = \left(l^i_{jk} x^j \right)$ is itself a Nijenhuis operator that is called the *linearisation* of L at the point $p \in M$.

Question. Are there any special properties of Nijenhuis operators $L_{\mathrm{lin}} = \left(l^i_{jk} x^j \right)$ whose components are linear in local coordinates?

Answer. The corresponding tensor l^i_{jk} defines a structure of a left-symmetric algebra $(\mathfrak{a}_L, *)$ on $T_p M$, and the converse is also true.

Comparison with Poisson geometry

Let $P = \left(P^{ij}(x)\right)$ be a Poisson structure, $x \in \mathbb{R}^n$
If $P(0) = 0$, then

$$P^{ij}(x) = 0 + P^{ij}_k x^k + \dots$$

where the linear part $P_{\mathrm{lin}} = \left(P^{ij}_k x^k\right)$ is a Lie-Poisson structure, i.e., P^{ij}_k form a structure tensor of a certain Lie algebra.
Conversely, if $\mathfrak{g}$ is a Lie algebra then $\mathfrak{g}^*$ carries a natural Poisson structure (Poisson tensor)

$$P_{\mathfrak{g}} = \left(c^k_{ij} x_k\right)$$

Linearisation of a Poisson structure $=$ Lie algebra

or, more or less equivalently,

Linear Poisson structures $=$ Lie-Poisson structures

Left-symmetric algebras

Recall that an algebra $\mathfrak{a}$, in a very general context, is a vector space with a bilinear operation $* : \mathfrak{a} \times \mathfrak{a} \to \mathfrak{a}$. Different types of algebras: commutative, associative, unital, Lie, Frobenius, etc.

Left-symmetric algebras can be understood as a generalisation of associative algebras. Namely, the associativity condition[1]

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = 0$$

is replaced with a weaker condition as follows.

Definition

An algebra $(\mathfrak{a}, *)$ is called *left-symmetric* if:

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = \eta * (\xi * \zeta) - (\eta * \xi) * \zeta,$$

for all $\xi, \eta, \zeta \in \mathfrak{a}$.

In particular, every associative algebra is left-symmetric. But there are many other examples.

[1] The left-hand side $\mathcal{A}(\xi, \eta, \zeta) = \xi * (\eta * \zeta) - (\xi * \eta) * \zeta$ is called *associator*.

More examples

Example

Consider the two-dimensional algebra $\mathfrak{a} = \mathrm{Span}(e_1, e_2)$ with the relations

$$\begin{aligned} e_1 * e_1 &= e_2 \\ e_1 * e_2 &= 0 \\ e_2 * e_1 &= -e_1 \\ e_2 * e_2 &= -2e_2 \end{aligned}$$

This algebra is left-symmetric, but not associative.

Example

Consider functions f on the real line $\mathbb{R}$ with coordinate x and introduce the following operation

$$f * g = f\, g_x,$$

where g_x denotes the derivative in x. The associator

$$\begin{aligned} &f * (g * h) - (f * g) * h = f * (g\, h_x) - (f\, g_x) * h = \\ &= f\, g_x h_x + fg\, h_{xx} - f\, g_x h_x = fg\, h_{xx}. \end{aligned}$$

is symmetric w.r.t. f and g. Obviously, this operation is not associative, but left-symmetric.

More examples

Example

Let M be a manifold with a flat symmetric connection ∇ and define the operation on vector fields as

$$\xi * \eta = \nabla_\xi \eta.$$

The associator takes the form

$$\xi * (\eta * \zeta) - (\xi * \eta) * \zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_{\nabla_\xi \eta} \zeta.$$

The condition $\mathcal{A}(\xi, \eta, \zeta) = \mathcal{A}(\eta, \xi, \zeta)$ takes the form

$$\begin{aligned}\nabla_\xi \nabla_\eta \zeta - \nabla_{\nabla_\xi \eta} \zeta - \nabla_\eta \nabla_\xi \zeta + \nabla_{\nabla_\eta \xi} \zeta = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{(\nabla_\eta \xi - \nabla_\xi \eta)} \zeta = \\ = \nabla_\xi \nabla_\eta \zeta - \nabla_\eta \nabla_\xi \zeta - \nabla_{[\xi, \eta]} \zeta = 0.\end{aligned}$$

Relationship between LSAs and Nijenhuis operators.

Theorem (Winterhalder)

An operator of the form $L = \left(l^i_{jk} x^j \right)$ is Nijenhuis if and only if l^i_{jk} form the structure constants of a left-symmetric algebra, i.e., the operation

$$\xi * \eta = \sum_{i,j,k} l^i_{jk} \xi^j \eta^k e_i, \quad \xi = \xi^j e_j,\ \eta = \eta^k e_k,$$

defines the structure of a left symmetric algebra on the vector space $\mathfrak{a}$ with a basis $e_1, \dots, e_n$.

Two slightly different versions of this theorem

Let R be a Nijenhuis operator such that $R(p) = 0$. Then the tangent space T_pM can be endowed with an LSA structure.

Proposition

Take tangent vectors $\xi_0, \eta_0 \in T_pM$ and introduce the following operation on T_pM:

$$\xi_0 * \eta_0 = [R\xi, \eta](p),$$

where ξ and η are (arbitrary) vector fields such that $\xi(p) = \xi_0$, $\eta(p) = \eta_0$. This operation is well defined and defines an LSA structure on T_pM.

Conversely, every left-symmetric algebra $(\mathfrak{a}, *)$ carries a Nijenhuis structure on itself.
Consider $\mathfrak{a}$ as an affine space (manifold). The tangent space $T_\xi\mathfrak{a}$, $\xi \in \mathfrak{a}$, is naturally identified with $\mathfrak{a}$ itself.

Proposition

Let $R : T_\xi\mathfrak{a} \to T_\xi\mathfrak{a}$ be defined by

$$R(\eta) = R_\xi(\eta) = \eta * \xi.$$

Then R is a Nijenhuis operator on $\mathfrak{a}$.

Proof of the first Proposition

By definition, $\xi_0 * \eta_0 = [R(\xi), \eta](p)$.

For ξ_0, η_0, ζ_0 we compute

$$\mathcal{A}(\xi_0, \eta_0, \zeta_0) - \mathcal{A}(\eta_0, \xi_0, \zeta_0) =$$

$$\xi_0 * (\eta_0 * \zeta_0) - (\xi_0 * \eta_0) * \zeta_0 - \Big(\eta_0 * (\xi_0 * \zeta_0) - (\eta_0 * \xi_0) * \zeta_0\Big) =$$

$$\big[R\xi, [R\eta, \zeta]\big] - \big[R\,[R\xi, \eta], \zeta\big] - \big[R\eta, [R\xi, \zeta]\big] + \big[R\,[R\eta, \xi], \zeta\big]\,|_{\text{at point } p} =$$

(using the anti-symmetry and Jacobi identity for the Lie bracket of vector fields)

$$\big[[R\xi, R\eta], \zeta\big] - \big[R\,[R\xi, \eta], \zeta\big] + \big[R\,[R\eta, \xi], \zeta\big]\,|_{\text{at point } p} =$$

$$\big[[R\xi, R\eta] - R\,[R\xi, \eta] - R\,[\xi, R\eta], \zeta\big]\,|_{\text{at point } p} =$$

(using the fact that $R(p) = 0$)

$$\big[R^2[\xi, \eta] + [R\xi, R\eta] - R\,[R\xi, \eta] - R\,[\xi, R\eta], \zeta\big]\,|_{\text{at point } p} =$$

(and finally using the fact that R is Nijenhuis)

$$[0, \zeta] = 0.$$

One property of left-symmetric algebras

Recall the following fundamental property of associative algebras.

Proposition

Every associative algebra $(\mathfrak{a}, *)$ carries a natural Lie algebra structure $(\mathfrak{a}, [\ ,\ ])$, namely
$$[\xi, \eta] = \xi * \eta - \eta * \xi.$$

In fact, the associativity condition can be relaxed.

Proposition

Every left-symmetric algebra $(\mathfrak{a}, *)$ carries a natural Lie algebra structure $(\mathfrak{a}, [\ ,\ ])$, namely
$$[\xi, \eta] = \xi * \eta - \eta * \xi.$$

For this reason, left-symmetric algebras are also known under the name *pre-Lie algebras.*

Proof

The operation is bilinear and skew-symmetric, thus, the only thing we need to prove is the Jacobi identity. For arbitrary triple $\xi, \eta, \zeta \in \mathfrak{a}$ we have

$$
\begin{aligned}
&[\xi,[\eta,\zeta]] + [\eta,[\zeta,\xi]] + [\zeta,[\xi,\eta]] = \\
=&[\xi, \eta * \zeta - \zeta * \eta] + [\eta, \zeta * \xi - \xi * \zeta] + [\zeta, \xi * \eta - \eta * \xi] = \\
=&\xi * (\eta * \zeta) - \xi * (\zeta * \eta) - (\eta * \zeta) * \xi + (\zeta * \eta) * \xi + \\
+&\eta * (\zeta * \xi) - \eta * (\xi * \zeta) - (\zeta * \xi) * \eta + (\xi * \zeta) * \eta + \\
+&\zeta * (\xi * \eta) - \zeta * (\eta * \xi) - (\xi * \eta) * \zeta + (\eta * \xi) * \zeta = \\
=&\mathcal{A}(\xi,\eta,\zeta) - \mathcal{A}(\xi,\zeta,\eta) + \mathcal{A}(\eta,\zeta,\xi) - \mathcal{A}(\eta,\xi,\zeta) + \\
+&\mathcal{A}(\zeta,\xi,\eta) - \mathcal{A}(\zeta,\eta,\xi)
\end{aligned}
$$

We see, that the Jacobi condition is an alternated sum of associators (this holds for arbitrary algebra). Thus, the left-symmetry (as well as right-symmetry or symmetry in first and third argument) yields zero: the corresponding terms cancel out.

Linearisation and non-degeneracy problems

Definition

Let L be a Nijenhuis operator and $L(p) = 0$ so that $L(x) = L_{\mathrm{lin}}(x) + \dots$, where L_{lin} is a linear part of L.

We will say that L is *linearisable* at p if there exists a coordinate transformation that reduces L to its linear part L_{lin}.

Linearisation problem. Given L such that $L(p) = 0$, find out whether L is linearisable or not?

Definition

A left-symmetric algebra $\mathfrak{a}$ is called *non-degenerate* if any Nijenhuis operator L, whose linearisation 'coincides' with $\mathfrak{a}_L$ at a singular point $p \in M$, is linearisable at this point.

Non-degeneracy problem. Describe all non-degenerate left-symmetric algebras.

Comparison with Poisson geometry 2

The above two problems are copy-pasted from Poisson geometry. Some well known facts:

- In dimension 2, let $\{x, y\} = f(x, y)$, with $f(0, 0) = 0$, $\mathrm{d}\, f(0, 0) \neq 0$. Then this Poisson structure is linearisable, i.e., there exist local coordinates $\tilde{x}, \tilde{y}$ such that $\{\tilde{x}, \tilde{y}\} = \tilde{y}$. In other words, the non-commutative 2-dim Lie algebra (defined by the relation $[e_1, e_2] = e_2$) is non-degenerate.
- Let P be a (smooth) Poisson structure in $\mathbb{R}^3(x, y, z)$ such that

$$\{x, y\} = z + \dots, \quad \{y, z\} = x + \dots, \quad \{z, x\} = y + \dots$$

  where 'dots' denote higher order terms. Then there exists another coordinate system $\tilde{x}, \tilde{y}, \tilde{z}$ such that

$$\{\tilde{x}, \tilde{y}\} = \tilde{z}, \quad \{\tilde{y}, \tilde{z}\} = \tilde{x}, \quad \{\tilde{z}, \tilde{x}\} = \tilde{y}.$$

  In other words, P is always linearisable, i.e., the Lie algebra $so(3)$ is non-degenerate (in the smooth sense).

Comparison with Poisson geometry 2 continued...

- Let P be a (smooth) Poisson structure in $\mathbb{R}^3(x, y, z)$ such that

$$\{x, y\} = 2y + \dots, \quad \{x, z\} = -2z + \dots, \quad \{y, z\} = x + \dots$$

where 'dots' denote higher order terms. Then there exists another coordinate system $\tilde{x}, \tilde{y}, \tilde{z}$ such that

$$\{\tilde{x}, \tilde{y}\} = 2\tilde{y}, \quad \{\tilde{x}, \tilde{z}\} = -2\tilde{z}, \quad \{\tilde{y}, \tilde{z}\} = \tilde{x}.$$

In other words, P is linearisable, i.e., the Lie algebra $sl(2)$ is non-degenerate in the real analytic sense (but not in the smooth sense!).

- Every semisimple Lie algebra is non-degenerate in the real analytic sense (bit not necessarily in the smooth sense).
- Every compact Lie algebra is non-degenerate in the smooth sense.

Classification of LSAs in dimension one

Theorem (Exercise)

There are two non-isomorphic left-symmetric algebras $\mathfrak{a} = \mathrm{Span}(\eta)$ in dimension 1 defined by the relations:

1. $\eta * \eta = 0$ (trivial algebra);
2. $\eta * \eta = \eta$ (non-trivial algebra)

Indeed, a one-dimensional algebra is defined by one single relation

$$\eta * \eta = a\eta, \quad a \in \mathbb{R}.$$

If $a \neq 0$, in can be made equal to 1 by rescaling $\eta \mapsto \frac{1}{a}\eta$.

Classification theorem in dim = 2

Theorem

Up to isomorphism there are two continuous families and 10 exceptional two dimensional real left-symmetric algebras. The complete list of normal forms is presented in Table 1 and Table 2 below. For every algebra we give

- *All non-zero structure relations for a given basis* η_1, η_2
- *The operator* $R = R_\eta$ *of right multiplication by* $\eta = x\eta_1 + y\eta_2$ *(this is a Nijenhuis operator!)*
- *The operator* $L = L_\eta$ *of left multiplication by* $\eta = x\eta_1 + y\eta_2$.

The letter $\mathfrak{b}$ *stands for algebras with non-abelian associated Lie algebra and* $\mathfrak{c}$ *for algebras with Abelian associated Lie algebra.*

Classification theorem: table 1

Name	Structure relations	L	R
$\mathfrak{b}_{1,\alpha}$	$\eta_2 * \eta_1 = \eta_1,$ $\eta_2 * \eta_2 = \alpha\eta_2$	$\begin{pmatrix} y & 0 \\ 0 & \alpha y \end{pmatrix}$	$\begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$
$\mathfrak{b}_{2,\beta}, \beta \neq 1$	$\eta_1 * \eta_2 = \eta_1,$ $\eta_2 * \eta_1 = \beta\eta_1$ $\eta_2 * \eta_2 = \eta_2$	$\begin{pmatrix} \beta y & x \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} y & \beta x \\ 0 & y \end{pmatrix}$
$\mathfrak{b}_3$	$\eta_2 * \eta_1 = \eta_1,$ $\eta_2 * \eta_2 = \eta_1 + \eta_2$	$\begin{pmatrix} y & y \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} 0 & x + y \\ 0 & y \end{pmatrix}$
$\mathfrak{b}_4^+$	$\eta_1 * \eta_1 = \eta_2,$ $\eta_2 * \eta_1 = -\eta_1$ $\eta_2 * \eta_2 = -2\eta_2$	$\begin{pmatrix} -y & 0 \\ x & -2y \end{pmatrix}$	$\begin{pmatrix} 0 & -x \\ x & -2y \end{pmatrix}$
$\mathfrak{b}_4^-$	$\eta_1 * \eta_1 = -\eta_2,$ $\eta_2 * \eta_1 = -\eta_1$ $\eta_2 * \eta_2 = -2\eta_2$	$\begin{pmatrix} -y & 0 \\ -x & -2y \end{pmatrix}$	$\begin{pmatrix} 0 & -x \\ -x & -2y \end{pmatrix}$
$\mathfrak{b}_5$	$\eta_1 * \eta_2 = \eta_1,$ $\eta_2 * \eta_2 = \eta_1 + \eta_2$	$\begin{pmatrix} 0 & x + y \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} y & y \\ 0 & y \end{pmatrix}$

Classification theorem: table 2

Name	Structure relations	$L = R$
$\mathfrak{c}_1$		$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$\mathfrak{c}_2$	$\eta_2 * \eta_2 = \eta_2$	$\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$
$\mathfrak{c}_3$	$\eta_2 * \eta_2 = \eta_1$	$\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$
$\mathfrak{c}_4$	$\eta_2 * \eta_2 = \eta_2$ $\eta_2 * \eta_1 = \eta_1$ $\eta_1 * \eta_2 = \eta_1$	$\begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$
$\mathfrak{c}_5^+$	$\eta_2 * \eta_2 = \eta_2$ $\eta_2 * \eta_1 = \eta_1$ $\eta_1 * \eta_2 = \eta_1$ $\eta_1 * \eta_1 = \eta_2$	$\begin{pmatrix} y & x \\ x & y \end{pmatrix}$
$\mathfrak{c}_5^-$	$\eta_2 * \eta_2 = \eta_2$ $\eta_2 * \eta_1 = \eta_1$ $\eta_1 * \eta_2 = \eta_1$ $\eta_1 * \eta_1 = -\eta_2$	$\begin{pmatrix} y & x \\ -x & y \end{pmatrix}$

Example of a non-degenerate LSA

Proposition

The left-symmetric algebra $\mathfrak{c}_5^+$ is non-degenerate. Equivalently, if $L = L(x, y)$ is a Nijenhuis operator of the form

$$L = \begin{pmatrix} y & x \\ x & y \end{pmatrix} + \text{higher order} \geq 2 \text{ terms} ,$$

Then there exists a smooth coordinate change, centred at 0 that transforms L into its linear part.

Proof. Let $f = \operatorname{tr} L$ and $g = \det L$. Then

$$f = 2y + \dots ,$$
$$g = y^2 - x^2 + \dots .$$

Here $\dots$ stand for terms of orders ≥ 2 and ≥ 3 respectively. Instead of reducing L to the required form, we will be looking for a suitable coordinate transformations to simplify f and g.

Proof of Proposition (continued...)

Step 1. As $f = \operatorname{tr} L = 2y + \dots$, we may set $y_{\rm new} = \frac{1}{2} f$ so that $f = \operatorname{tr} L = 2y_{\rm new}$. The first coordinate remains unchanged. Keeping the same notation x, y for the new coordinates, we now have

$$\begin{aligned} f &= 2y \\ g &= y^2 - x^2 + \dots . \end{aligned}$$

Step 2. To simplify g, we treat y as a parameter and apply the parametric Morse lemma, which says that we can introduce a new variable $x_{\rm new} = h(x, y) = x + \dots$ (y remains unchanged) in such a way that

$$g = -x_{\rm new}^2 + k(y), \quad \text{with } k(y) = y^2 + \dots$$

Keeping the same notation x, y for the new coordinates, we now have

$$\begin{aligned} f &= 2y \\ g &= -x^2 + k(y) \end{aligned}$$

where f and g are the trace and determinant of our Nijenhuis operator L.

Question. What can we say about L in this situation?

Proof of Proposition (continued...)

Recall that a Nijenhuis operator can be reconstructed from the coefficients of the characteristic polynomial by using the following fundamental formula

$$L = J^{-1}\begin{pmatrix} \sigma_1 & 1 \\ \sigma_2 & 0 \end{pmatrix} J, \quad \text{where } J = \left(\frac{\partial(\sigma_1, \sigma_2)}{\partial(x, y)}\right),$$

and $\sigma_1 = \operatorname{tr} L, \sigma_2 = -\det L$ are the coefficients of the characteristic polynomial of L. A simple computation gives the following formula for L

$$L = \begin{pmatrix} \frac{1}{2}k' & x + \frac{k'(4y-k')-4k}{4x} \\ x & 2y - \frac{1}{2}k' \end{pmatrix}$$

Now another miracle in Nijenhuis geometry... Look at the fraction $\frac{k'(4y-k')-4k}{4x}$. Its numerator is a function of y only. Hence this fraction is smooth at the origin if and only if $k'(4y-k')-4k \equiv 0$. It shows that $k = k(y)$ must be very special!

This relation (after differentiating by y) implies $k''(y)(2y-k') = 0$ and since $k''(y) \neq 0$, we get $k' = 2y$ and $k = y^2$, giving finally $L = \begin{pmatrix} y & x \\ x & y \end{pmatrix}$, as required.

Example of a degenerate LSA

Proposition

The left-symmetric algebra $\mathfrak{c}_4$ is degenerate.
More specifically, the following non-linear perturbation

$$\begin{pmatrix} y & x \\ 0 & y \end{pmatrix} \quad \mapsto \quad L = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix} + \begin{pmatrix} yx^2 & x^3 \\ -xy^2 & -yx^2 \end{pmatrix}$$

gives a non-linearisable Nijenhuis operator L.

Proof. Consider the trace and determinant of L:

$$\operatorname{tr} L = 2y, \quad \det L = y^2 + y^2x^2$$

Obviously, the discriminant of the characteristic polynomial $\chi_L(t)$ is not identically zero, so that generically L has two different eigenvalues, while its linear part $L_{\mathrm{lin}} = \begin{pmatrix} y & x \\ 0 & y \end{pmatrix}$ has one single eigenvalue y of multiplicity 2.

Hence, L and L_{lin} are essentially different and cannot be reduced to each other by a coordinate transformation.

Another classification theorem

Theorem (Smooth case)

In the smooth category

Degenerate LSA	*Non-degenerate LSA*
$\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4,$ $\mathfrak{b}_5, \mathfrak{b}_{2,\beta}$ $\mathfrak{b}_{1,\alpha}$ for $\alpha \in \Sigma_{\mathrm{sm}}$	$\mathfrak{b}_4^+, \mathfrak{b}_4^-, \mathfrak{c}_5^+, \mathfrak{c}_5^-$ $\mathfrak{b}_3, \mathfrak{b}_{1,\alpha}$ for $\alpha \notin \Sigma_{\mathrm{sm}}$

Theorem (Analytic case)

In the real analytic category

Degenerate LSA	*Non-degenerate LSA*
$\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4,$ $\mathfrak{b}_5, \mathfrak{b}_{2,\beta}$ $\mathfrak{b}_{1,\alpha}$ for $\alpha \in \Sigma_{\mathrm{an}}$	$\mathfrak{b}_4^+, \mathfrak{b}_4^-, \mathfrak{c}_5^+, \mathfrak{c}_5^-$ $\mathfrak{b}_3, \mathfrak{b}_{1,\alpha}$ for $\alpha \notin \Sigma_{\mathrm{an}}$

Comments on the sets Σ_{sm} and Σ_{an} related to $\begin{pmatrix} 0 & x \\ 0 & \alpha y \end{pmatrix}$

The continuous fraction for α is a decomposition of α in the form

$$\alpha = q_0 + \frac{1}{q_1 + \frac{1}{q_2 + \dots}},$$

where $q_0 \in \mathbb{Z}$ and $q_i, i \geq 1$ are in $\mathbb{N}$. Let $[q_0, q_1, q_2, \dots]$ be a decomposition of an irrational α into the continuous fraction. If the series

$$B(x) = \sum_{i=0}^{\infty} \frac{\log q_{i+1}}{q_i}$$

converges, then α is a Brjuno number.

- Σ_{sm} contains $\alpha < 0$, $\alpha = \frac{1}{m}$ for $m \geq 2$ and s for $s \geq 3$.
- Σ_{an} contains $\alpha = -\frac{p}{q}$, negative irrational numbers that are not Brjuno numbers, $\alpha = \frac{1}{m}$ for $m \geq 2$ and s for $s \geq 3$.

Non-degeneracy of the diagonal algebra

Theorem (Real analytic or formal)

Let $L(x) = L_{\text{lin}}(x) + L_2(x) + L_3(x) + \dots$ *with*

$$L_{\text{lin}}(x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}.$$

Then $L(x)$ *is linearisable. In other words, the diagonal left-symmetric algebra is non-degenerate.*

Differentially non-degenerate LSAs

Definition

We say that a left-symmetric algebra $\mathfrak{a}$ is *differentially non-degenerate* if the Nijenhuis operator R_ξ of right multiplication (see above) is differentially non-degenerate (at a generic point $\xi \in \mathfrak{a}$).

Recall that the entries of the operator $R_\xi = \left(R^i_j(\xi)\right)$ are linear functions in ξ, i.e., $R^i_j = \sum_\alpha l^i_{j\alpha}\xi^\alpha$, where $\xi = \sum_\alpha \xi^\alpha e_\alpha$. This implies that the coefficient $\sigma_k(\xi)$ of the characteristic polynomial

$$\chi_{R_\xi}(t) = \det(t\,\mathrm{Id} - R_\xi) = t^n - \sigma_1(\xi)t^{n-1} - \sigma_2(\xi)t^{n-2} - \dots - \sigma_n(\xi)$$

is a homogeneous polynomial in $\xi^1, \dots, \xi^n$ of degree k.
The differential non-degeneracy condition means that the polynomials $\sigma_1, \dots, \sigma_n$ are algebraically independent.

Open problem 1. Classify/describe differentially non-degenerate left-symmetric algebras. (The problem is solved in dimensions 1,2,3.)

Open problem 2. Is it true that a differentially non-degenerate left-symmetric algebra is non-degenerate.

Purely algebraic statement of Open Problem 1.

Open problem 1′. Describe all collections of algebraically independent homogeneous polynomials $\sigma_1, \dots, \sigma_n$ in n variables $x_1, \dots, x_n$ ($\deg \sigma_k = k$) such that the entries of the matrix

$$R = \left(\frac{\partial \sigma_k}{\partial x^j}\right)^{-1} \begin{pmatrix} \sigma_1 & 1 & & \\ \sigma_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \left(\frac{\partial \sigma_k}{\partial x^j}\right)$$

are linear functions in $x^1, \dots, x^n$ (here $\left(\frac{\partial \sigma_k}{\partial x^j}\right)$ denotes the Jacobi matrix of the collection of polynomials $\sigma_1, \dots, \sigma_n$).

Comment. According to the fundamental property of Nijenhuis operators (see Lecture 1), R is Nijenhuis for any collection of independent polynomials $\sigma_1, \dots, \sigma_n$. But in general, the entries of R are rational functions of the form $R^i_j = \frac{P_{ij}(x)}{Q(x)}$ where $\deg P_{ij} = n + 1$, $\deg Q = n$ and $Q = \det\left(\frac{\partial \sigma_k}{\partial x^j}\right)$. Sometimes, a miracle happens: each P_{ij} turns out to be divisible by Q, and then R defines a left-symmetric algebra.

Exercises

- Prove that $\mathfrak{c}_5^+$ is isomorphic to the direct sum of two one-dimensional non-trivial algebras.
- Prove that $\mathfrak{c}_5^-$ is a real form of one-dimensional complex algebra with non-trivial multiplication.
- Classify of differentially non-degenerate LSAs in dimension 2 (without using the classification theorem for LSAs in 2D).
- Let $\mathfrak{g}$ be a Lie algebra. Consider the Lie-Poisson bracket $P_x = \left(c^i_{jk} x_i\right)$ on $\mathfrak{g}^*$ and assume that $\det P_a \neq 0$ at a generic point $a \in \mathfrak{g}^*$ (such $\mathfrak{g}$ is called a Frobenius Lie algebra). Let us introduce an operator (field of endomorphisms) R on g^* by setting

$$R(x) : T_x\mathfrak{g}^* \to T_x\mathfrak{g}^*, \quad R(x) = P_x \circ P_a^{-1}$$

where the P_x, P_a are understood as (skew-symmetric) linear maps $R_x, R_a : \mathfrak{g} = T_x^*\mathfrak{g}^* \to \mathfrak{g}^* = T_x\mathfrak{g}^*$. Prove that R is Nijenhuis operator on $\mathfrak{g}^*$ with linear entries, which implies that $\mathfrak{g}^*$ carries a structure of a left-symmetric algebra (this structure depends on the choice of a regular element $a \in \mathfrak{g}^*$).

Exercises

- Two Nijenhuis operators L_1 and L_2 are called compatible if their sum $L_1 + L_2$ is a Nijenhuis operator too.
  (a) Check that this condition implies that any linear combination $a_1 L_1 + a_2 L_2$ is Nijenhuis.
  (b) Write down the compatibility condition in tensorial form, like

  $$L_1 L_2 [u, v] - L_1 [L_2 u, v] - \cdots = 0 \qquad \text{for all vector fields } u, v$$

  The expression in the l.h.s. is known as Frölicher-Nijenhuis bracket of two operators.
  (c) (Argument shift method à la Mishchenko–Fomenko) Let R_ξ be the Nijenhuis operator associated with a left-symmetric algebra $\mathfrak{a}$ and R_a be the constant operator obtained by setting $\xi = a \in \mathfrak{a}$. Then R_ξ and R_a are compatible.