Nijenhuis Geometry and Applications
Lecture 1
Basic facts, singular points, gl-regular Nijenhuis operators

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Plan

- Historical remarks
- What is Nijenhuis Geometry?
- Motivation and research agenda
- Nijenhuis operators
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- Splitting theorem
- Jordan block: typical behaviour of Nijenhuis operators
- Regular Nijenhuis manifolds
- Generalised Newlander–Nirenberg theorem
- Some global results
- Exercises

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Albert Nijenhuis

Albert Nijenhuis (November 21, 1926 – February 13, 2015),

Dutch-American mathematician who specialised in differential geometry and the theory of deformations in algebra and geometry, and later worked in combinatorics.

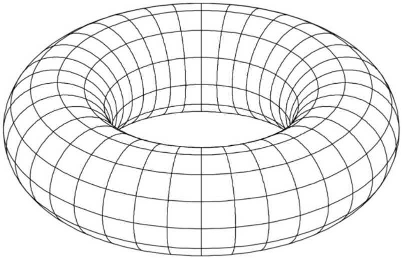
Alma mater: University of Amsterdam

Doctoral advisor: Prof. Jan Arnoldus Schouten

https://en.wikipedia.org/wiki/Albert_Nijenhuis

What is GEOMETRY?

Space, manifold M^n $+$ Structure



Structure is usually defined by means of a tensor, like,
g_{ij}, ω_{ij}, or P^{ij}

Naively, in coordinates, the geometric structure is defined by means of a matrix $A = \left(a_{ij}(x)\right)$ whose entries depend on coordinates $x = (x^1, \dots, x^n)$ and satisfy some algebraic and differential conditions.

Nijenhuis geometry. Our motivation

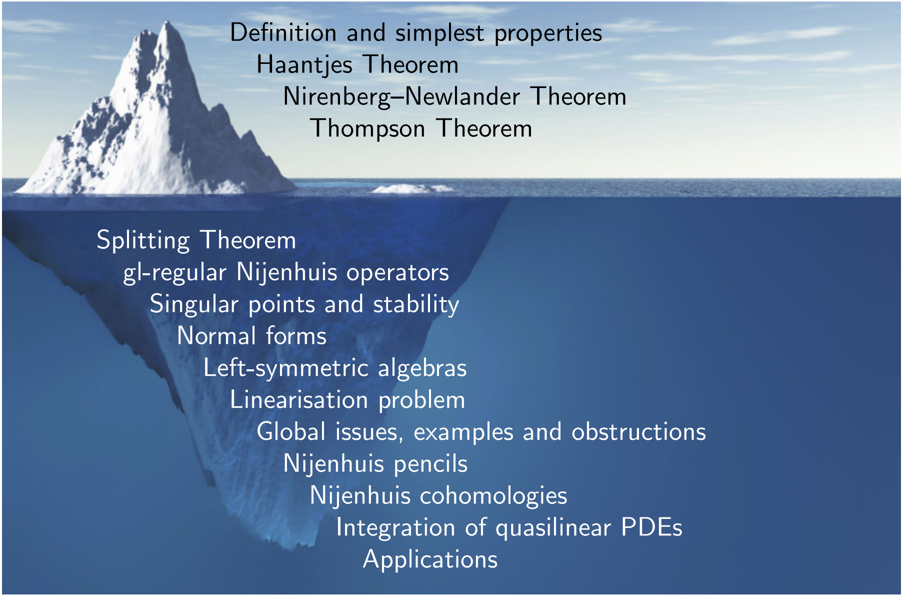
Definition

By Nijenhuis operators we understand $(1, 1)$-tensors $L = \left(L^i_j(x)\right)$ with vanishing Nijenhuis torsion. A manifold M endowed with such an operator it is called a Nijenhuis manifold.

Motivation

- Riemannian, Kähler, symplectic, Poisson... Nijenhuis geometry is the next, most natural candidate to continue this list.
- In the context of the bi-Hamiltonian formalism, Nijenhuis operators occur as recursion operators (for both finite- and infinite-dimensional cases like systems of hydrodynamic type and KdV equations).
- In the theory of integrable geodesic flows, projectively equivalent Riemannian metrics are related by means of a Nijenhuis operator.
- In topology of integrable systems, singularities of Lagrangian fibrations related to bi-Hamiltonian systems correspond to singular points of the corresponding Nijenhuis recursion operators
- In integrable systems on Lie algebras, the algebraic Nijenhuis operators are used in the study of Lie-Poisson pencils.

Nijenhuis Geometry



Definition and simplest properties
Haantjes Theorem
Nirenberg–Newlander Theorem
Thompson Theorem

Splitting Theorem
gl-regular Nijenhuis operators
Singular points and stability
Normal forms
Left-symmetric algebras
Linearisation problem
Global issues, examples and obstructions
Nijenhuis pencils
Nijenhuis cohomologies
Integration of quasilinear PDEs
Applications

Research agenda

The ultimate goal of our research programme is to answer three fundamental questions:

(A) Local description: to what form can one bring a Nijenhuis operator near almost every point by a local coordinate change?

(B) Singular points: what does it mean for a point to be generic or singular in the context of Nijenhuis geometry? What singularities are non-degenerate/stable? How do Nijenhuis operators behave near non-degenerate and stable singular points?

(C) Global properties: what restrictions on a Nijenhuis operator are imposed by the topology of the underlying manifold? And conversely, what are topological obstructions to a Nijenhuis manifold carrying a Nijenhuis operator with specific properties?

as well as to work on

(D) Applications of Nijenhuis Geoetry: in geometry, algebra and mathematical physics

Nijenhuis operators

Let L be a $(1,1)$-tensor field (operator) on a smooth manifold M. The *Nijenhuis torsion* $\mathcal{N}_L$ of the operator L is a $(1,2)$-tensor that can be defined in several equivalent ways.

- As a vector-valued 2-form:

$$\mathcal{N}_L(\xi,\eta) = L^2[\xi,\eta] + [L\xi, L\eta] - L[L\xi,\eta] - L[\xi, L\eta].$$

- As a map from "vector fields" to "endomorphisms":

$$\mathcal{N}_L : \ \xi \mapsto L\mathcal{L}_\xi L - \mathcal{L}_{L\xi} L.$$

- As a map from "1-forms" to "2-forms":

$$\mathcal{N}_L : \ \alpha \mapsto \beta, \quad \text{where}$$

$$\beta(\cdot\,,\cdot) = \mathrm{d}\,(L^{*2}\alpha)(\cdot\,,\cdot) + \mathrm{d}\,\alpha(L\cdot\,, L\cdot) - \mathrm{d}\,(L^*\alpha)(L\cdot\,,\cdot) - \mathrm{d}\,(L^*\alpha)(\cdot\,, L\cdot).$$

- In local coordinates:

$$(\mathcal{N}_L)^i_{jk} = L^l_j \frac{\partial L^i_k}{\partial x^l} - L^l_k \frac{\partial L^i_j}{\partial x^l} - L^i_l \frac{\partial L^l_k}{\partial x^j} + L^i_l \frac{\partial L^l_j}{\partial x^k}.$$

Definition. If $\mathcal{N}_L \equiv 0$, then L is called a *Nijenhuis operator*.

Elementary examples

- Constant operator:
$$L(x) = \left(L^i_j\right)$$
with L^i_j being constant for all i, j
- Scalar operator:
$$L(x) = f(x) \cdot \mathrm{Id},$$
where $f(x)$ is an arbitrary smooth function
- Complex structure
- $L(x) = \begin{pmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{pmatrix}$
- $L(x) = \begin{pmatrix} x_5 & x_4 & x_3 & x_2 & x_1 \\ & x_5 & x_4 & x_3 & x_2 \\ & & x_5 & x_4 & x_3 \\ & & & x_5 & x_4 \\ & & & & x_5 \end{pmatrix}$

Generic and singular points

Definition

- A point $p \in M$ is called *algebraically generic*, if the algebraic type of L does not change in some neighbourhood $U(p) \subset M$.
- A point $p \in M$ is called *singular*, if it is not algebraically generic.
- A point $p \in M$ is called *differentially non-degenerate*, if the differentials $\mathrm{d}\,\sigma_1(x), \dots, \mathrm{d}\,\sigma_n(x)$ of the coefficients of the characteristic polynomial of $L(x)$ are linearly independent at this point.
- A singular point $p \in M$ is called *(C^k-) stable*, if for any perturbation

$$L(x) \quad \mapsto \quad \widetilde{L}(x) = L(x) + R_k(x)$$

such that $\widetilde{L}(x)$ is Nijenhuis and $R_k(x)$ has zero of order k at the point $p \in M$, there exists a local diffeomorphism $\phi : U(p) \to \widetilde{U}(p)$, $\phi(p) = p$, that transforms $L(x)$ to $\widetilde{L}(x)$.

Splitting theorem

Let $\chi_{L(x)}(t) = \det\big(t\,\mathrm{Id} - L(x)\big) = t^n - \sigma_1(x)t^{n-1} - \sigma_2(x)t^{n-2} - \ldots - \sigma_n(x)$ be the characteristic polynomial of L and

$$\chi_{L(p)}(t) = \chi_1(t)\,\chi_2(t)$$

be its factorisation at a point $p \in M^n$ into two factors with no common roots (over $\mathbb{R}$), $\deg \chi_1 = r$, $\deg \chi_2 = n - r$.

Theorem

There exist local coordinates $(x_1, \ldots, x_r, y_{r+1}, \ldots, y_n)$ such that

$$L(x, y) = \begin{pmatrix} L_1(x) & 0 \\ 0 & L_2(y) \end{pmatrix}, \tag{1}$$

with $\chi_1 = \chi_{L_1}$ and $\chi_2 = \chi_{L_2}$.

In particular the distributions $\mathcal{D}_i = \mathrm{Ker}\,\chi_i(L)$ $(i = 1, 2)$ are integrable.

In other words, L splits into a direct sum of two Nijenhuis operators: $L(x, y) = L_1(x) \oplus L_2(y)$.

Proof of Splitting Theorem

Step 1. If L is Nijenhuis, then L^2 is Nijenhuis too.

Step 2. If L is Nijenhuis, then $p(L)$ is Nijenhuis, where $p(\cdot)$ is a polynomial (with constant coefficients).

Step 3. If L is Nijenhuis, then any convergent power series $\sum a_k L^k$, $a_k \in \mathbb{R}$, i.e. any real analytic function $f(L)$ is Nijenhuis, e.g. $\exp L$, $\sin L$, ...

Step 4. Now let $T_x M = \mathcal{D}_1 \oplus \mathcal{D}_2$ be the decomposition of the tangent space into the subspaces related to the factorisation $\chi_L = \chi_1 \cdot \chi_2$ and $P_i : T_x M \to \mathcal{D}_i$ denote the projector onto $\mathcal{D}_i$.
Fact from Matrix Analysis: $P_i = \sum a_k L^k = f_i(L)$ real analytic function. Therefore P_i is Nijenhuis.

Step 5. P_i is a very simple operator with two constant eigenvalues, 0 and 1. We prove the Splitting Theorem for P_i instead, to find coordinates $(x_1, \dots, x_r, y_{r+1}, \dots, y_n)$.

Step 6. Then $L = LP_1 + LP_2 = \begin{pmatrix} L_1 & \\ & 0 \end{pmatrix} + \begin{pmatrix} 0 & \\ & L_2 \end{pmatrix}$, where LP_1 and LP_2 are both Nijenhuis.

Step 7. Verification that $\partial_{y_\alpha} L_1 = 0$ and $\partial_{x_\beta} L_2 = 0$ is straightforward from the definition.

Corollaries: Haantjes theorem

Corollary

Every Nienhuis operator L locally splits into a direct sum of Nijenhuis operators $L = L_1 \oplus L_2 \oplus \dots$ each of which at the point $p \in M$ has either a single real eigenvalue or a single pair of complex eigenvalues.

Theorem (Haantjes)

Let L be a Nijenhuis operator which is $\mathbb{R}$-diagonalisable at a point p and, all of its eigenvalues are different. Then there exist local coordinates $(x_1, \dots, x_n)$ such that

$$L(x) = \begin{pmatrix} \lambda_1(x_1) & & & \\ & \lambda_2(x_2) & & \\ & & \ddots & \\ & & & \lambda_n(x_n) \end{pmatrix}$$

Moreover, if $\lambda_i' \neq 0$, $i = 1, \dots, n$, then we can take the eigenvalues $u_i = \lambda_i(x_i)$ as new local coordinates to simplify L even further:

$$L(u) = \begin{pmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{pmatrix}$$

Jordan block: typical behaviour of Nijenhuis operators

Let $p \in M$ be a singular point for a Nijenhuis operator L. For example, $L(p)$ is conjugate to a Jordan block but it is not true any more at neighbouring points.

Theorem

Assume that L is differentially non-degenerate at a point $p \in M$. Then there exists a local coordinate system $x_1, \dots, x_n$ in which L takes the following canonical form:

$$L = \begin{pmatrix} x_1 & 1 & & & \\ x_2 & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ x_{n-1} & 0 & \dots & 0 & 1 \\ x_n & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2}$$

Corollary

Differentially non-degenerate singular points are C^2-stable.

Sketch of proof

Step 1. Important identity (follows from Definition 2: $L\mathcal{L}_\xi L - \mathcal{L}_{L\xi} L = 0$.)

$$\mathcal{L}_{L\xi}(\det L) = \det L \cdot \mathcal{L}_\xi \operatorname{tr} L$$

Step 2. Equivalently,

$$L^* \,\mathrm{d}(\det L) = \det L \cdot \mathrm{d} \operatorname{tr} L.$$

and, more generally, replacing L with $L - t\,\mathrm{Id}$:

$$(L - t\,\mathrm{Id})^* \,\mathrm{d}\,\chi_L(t) = \chi_L(t) \cdot \mathrm{d} \operatorname{tr} L$$

Here $\chi_L(t) = \det(t\,\mathrm{Id} - L) = t^n - \sigma_1 t^{n-1} - \dots - \sigma_n$.

Step 3. In matrix form, it becomes the following fundamental identity in Nijenhuis Geometry:

$$J\,L = S\,J, \quad \text{where } S = \begin{pmatrix} \sigma_1 & 1 & & \\ \sigma_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ \sigma_n & 0 & \dots & 0 \end{pmatrix} \text{ and } J = \left(\frac{\partial \sigma_i}{\partial x_j}\right)$$

Step 4. If L is differentially non-degenerate, i.e., $\sigma_1, \dots, \sigma_n$ are independent functions, we simply set $x_i = \sigma_i$.

Regular Nijenhuis manifolds

Definition

L is called $\mathfrak{gl}$-*regular*, if its $GL(n)$-orbit $\mathcal{O}(L) = \{X\,L\,X^{-1} \mid X \in GL(n)\}$ has maximal dimension, namely, $\dim \mathcal{O}(L) = n^2 - n$ (equivalently, each eigenvalue of L admits only one eigenvector).

A Nijenhuis manifold (M, L) is called *regular*, if $L(x)$ is $\mathfrak{gl}$-regular at each point $x \in M$.

Question: What is a local structure of a regular Nijenhuis manifold?

Theorem (Real analytic case)

There exists a local coordinate system such that

$$L = \begin{pmatrix} f_1(x) & 1 & & & \\ f_2(x) & 0 & 1 & & \\ \vdots & \vdots & \ddots & \ddots & \\ f_{n-1}(x) & 0 & \dots & 0 & 1 \\ f_n(x) & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The functions $f_1, \dots, f_n$ are not arbitrary but satisfy a PDE system:

$$F_{x_k} = L^{n-k} F_{x_n}, \qquad \textit{where } F = (f_1, f_2, \dots, f_n)^\top.$$

Problem of a Jordan block

Question. Let $L = J_0$ be a Jordan block at a point $p \in M$. What can we say about behaviour of L in a neighbourhood of p, if L is Nijenhuis? What are possible scenarios? Can we, for instance, "perturb" a Jordan block in such a way that exactly two eigenvalues appear with prescribed multiplicities?

Answer. All scenarios are allowed.

Consider the natural stratification of the set of regular operators

$$\mathrm{gl}(n,\mathbb{R})^{\mathrm{reg}} = \bigsqcup_{\sum k_s = n} W_{k_1,\dots,k_s}, \qquad k_1 \le \dots \le k_s,\ s \le n,\ k_i \in \mathbb{N},$$

where $W_{k_1,\dots,k_s}$ is the set of operators having s distinct eigenvalues with multiplicities $k_1,\dots,k_s$. Notice that the Jordan block belongs to the closure of each of $W_{k_1,\dots,k_s}$.

Theorem

For each stratum $W_{k_1,\dots,k_s} \subset \mathrm{gl}(n,\mathbb{R})$, there exists a Nijenhuis operator L in a neighborhood of $0 \in \mathbb{R}^n$ such that $L(0) = J_0$ and $L(x) \in \overline{W}_{k_1,\dots,k_s}$ for all $x \in U(0)$, where $\overline{W}_{k_1,\dots,k_s}$ is the closure of $W_{k_1,\dots,k_s}$ (either in the standard or Zariski topology).

Generalised Nirenberg-Newlander theorem

Theorem

Let L be a Nijenhuis operator on M with no real eigenvalues, i.e., its spectrum at every point $x \in M$ belongs to $\mathbb{C} \setminus \mathbb{R}$. Then

1. *M is a complex manifold w.r.t. a complex structure J canonically associated with L.*
2. *L is a complex holomorphic tensor field on M w.r.t. J, i.e. can be written in the form*
$$L^{\mathbb{C}} = \sum_{i,j=1}^{n} l^i_j(z)\, \mathrm{d}\, z^j \otimes \partial_{z^i}$$
with all the functions $l^i_j(z)$ being holomorphic in complex coordinates $z_1, \dots, z_n$.
3. *The complex Nijenhuis torsion of L vanishes, i.e.*
$$\left(\mathcal{N}_L^{\mathbb{C}}\right)^i_{jk} = l^m_j \frac{\partial l^i_k}{\partial z^m} - l^m_k \frac{\partial l^i_j}{\partial z^m} - l^i_m \frac{\partial l^m_k}{\partial z^j} + l^i_m \frac{\partial l^m_j}{\partial z^k} = 0.$$

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Key point: $J = f(L)$ where f is an analytic function on $\mathbb{C} \setminus \mathbb{R}$

Some global results

Theorem

Let L be a Nijenhuis operator on a closed connected manifold M with a non-real eigenvalue $\lambda \in \mathbb{C} \setminus \mathbb{R}$ at least at one point. Then this number λ is an eigenvalue of L with the same algebraic multiplicity at every point of M. Shortly: a Nijenhuis operator on a closed manifold may not have non-constant complex eigenvalues.

Corollary

A Nijenhuis operator L on a closed manifold cannot have differentially non-degenerate singular points (like e.g. 'standard' deformations of Jordan blocks).

Corollary

The eigenvalues of a Nijenhuis operator on the 4-dimensional sphere S^4 are all real.

Exercises

- Prove that the four definitions of Nijenhuis torsion/operator are equivalent.
- Using the third definition, prove the following important property (used for constructing many of integrable hierarchies). Let L be Nijenhuis and α_0 a closed differential 1-form such that $\alpha_1 = L^*\alpha_0$ is closed also. Then $\alpha_2 = L^*\alpha_1$, $\alpha_3 = L^*\alpha_2$, etc. are all closed.
- Prove that if the eigenvalues of 2×2 operator L are constant, then L is Nijenhuis.
- Prove the following necessary and sufficient condition in dimension 2: L is Nijenhuis if and only if $\mathrm{d} \det L = (\operatorname{adj} L)^* \, \mathrm{d} \operatorname{tr} L$. In more detail:

$$\Big((\det L)_x, (\det L)_y\Big) = \Big((\operatorname{tr} L)_x, (\operatorname{tr} L)_y\Big) \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \text{where } L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

- Use this criterion to check that $L = \begin{pmatrix} u & -v \\ v & u \end{pmatrix}$ with $u = u(x, y)$, $v = v(x, y)$ and $v \neq 0$, is Nijenhuis if and only if the function $u + \mathrm{i} v$ is holomorphic in complex variable $z = x + \mathrm{i} y$.

Exercises

- Prove that $L = \begin{pmatrix} x_1 & 1 & & \\ x_2 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ x_n & 0 & \dots & 0 \end{pmatrix}$ is Nijenhuis.
  Here $x_1, \dots, x_n$ are local coordinates.
- Prove that $L = \begin{pmatrix} 0 & 1 & & \\ 0 & 0 & \ddots & \\ \vdots & \vdots & \ddots & 1 \\ f_1 & f_2 & \dots & f_n \end{pmatrix}$ is Nijenhuis if and only if the differential 1-forms $\alpha = f_1 \, \mathrm{d}\, x_1 + f_2 \, \mathrm{d}\, x_2 + \dots + f_n \, \mathrm{d}\, x_n$ and $L^* \alpha$ are closed.

Thanks for your attention