Lecture 3.

Synchronous games.

$\mathcal{G} = (X, X, A, A, \lambda)$ is called synchronous if $\lambda(x, x, a, b) = 0$ whenever $a \neq b$.

Ex. 1. The colouring game:

G - a simple graph

$V(G) = X$ - vertices

A - colours

rule λ: to win: $x \sim y \Rightarrow a \neq b$

adjacent vertices

$\underline{x = x \Rightarrow a = b}$

synchronicity

2. Homomorphism game:

G, H - two simple graphs

$X = V(G)$, $A = V(H)$

rule λ: to win: $x \sim y \Rightarrow a \sim b$

$\underline{x = y \Rightarrow a = b}$

synchronicity

Def. A strategy / correlation $p = \{p(a, b \mid x, y)\}$ is called synchronous if $p(a, b \mid x, x) = 0$ whenever $a \neq b$

Notation: C_t^s, $t \in \{loc, q, qs, \bar{q}, qc\}$ for synchronous in class C_t.

Winning strategies for synchronous games must be synchronous

Synchronous correlation $\rightsquigarrow$ A C^*-algebra with trace.

Traces. 1. $M_n(\mathbb{C}) \ni a = (a_{ij})$

$$Tr(a) = \sum_1^n a_{ii}$$

- a - positive semi-definite $(a \geq 0)$

$$\Downarrow$$

$$Tr(a) \geq 0$$

- $Tr(ab) = Tr(ba)$
- $Tr(1_n) = n \quad \left(\frac{1}{n} Tr(1_n) = 1\right)$

Let A be a C^*-algebra.

A trace τ on A is a linear functional s.t.

1. τ is positive, i.e. $\tau(a) \geq 0$ if $a \geq 0$
2. $\tau(ab) = \tau(ba)$ $\forall a, b \in A$
3. $\tau(1) = 1$

1), 3) say that τ is a state on A.

A C^*-algebra may have a unique trace (like for $A = M_n(\mathbb{C})$, $\frac{1}{n} Tr$), many traces or no trace.

Ex. 1. $M_n(\mathbb{C}) \oplus M_m(\mathbb{C}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : \begin{array}{l} A \in M_n(\mathbb{C}) \\ B \in M_m(\mathbb{C}) \end{array} \right\}$

has traces

$$\tau_t \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right) = t \frac{1}{n} Tr A + (1-t) \frac{1}{m} Tr B$$

2. $B(\ell^2)$ has no trace $\rightsquigarrow$ Exercise

Look at $T: e_n \to e_{2n}$

$S: e_n \to e_{2n+1}$

write 1 as certain sum of TT^* and SS^*

and use traciality

What is the C^*-algebra related to synchr. correlations?

$p \in C_t \rightsquigarrow$ POVMs $\{E_{x,a}\}_{a\in A, x\in X} \overset{\text{Stinespring}}{\rightsquigarrow}$ Boca

PVMs $\{E_{x,a}\}_{a\in A}$, $x \in X$

i.e. projections $E_{x,a}$ with $\sum_a E_{x,a} = 1 \quad \forall x$

Consider $\ell^\infty_A = (\mathbb{C}^A, \|\cdot\|)$ generated by projections $e_a = (0,\dots,\underset{a}{1},0\dots0)$

and the free product

$\mathcal{A}_{X,A} := \underbrace{\ell^\infty_A *_1 \dots *_1 \ell^\infty_A}_{|X|\text{-copies}}$ $\leftarrow$ universal C^*-alg. generated by $\{e_{x,a}\}_a$, $x \in X$

$e_{x,a}$ $\leftarrow$ copy of e_a in x^{th} component

s.t. $\sum_a e_{x,a} = 1$

Universality in this case means that whenever $\{E_{x,a}\}_a$ - PVMs on H

$\exists!\ \pi$ - $*$-homomorphism $\mathcal{A}_{X,A} \to B(H)$

s.t. $\pi(e_{x,a}) = E_{x,a}$

Remark: It is convinient to think of ℓ^∞_A as $C^*(\mathbb{Z}_A)$ – the group C^*-alg of $\mathbb{Z}_A$ $\sim$ the cyclic group of order $|A|$

What is $C^*(G)$, G - discrete?

$\mathbb{C}[G]$ = vector space with basis $u_g, g \in G$

multiplication: $u_g \cdot u_h := u_{gh}$

which extends naturaly to $\mathbb{C}[G]$

involution: $(\sum \alpha_g u_g)^* := \sum \overline{\alpha_g}\, u_{g^{-1}}$

If $\rho : G \to B(H)$ is a unitary repr. it gives a $*$-repres. of $\mathbb{C}[G]$:

$\tilde{\rho} : \sum \alpha_g u_g \longmapsto \sum \alpha_g \rho(g)$

$C^*(G)$ is the completion of $\mathbb{C}[G]$
w.r.t. $\| \sum \alpha_g u_g \| = \sup_{\substack{\rho\text{-unitary} \\ \text{repr}}} \| \sum \alpha_g \rho(g) \|$

Any unitary repr. of $\mathbb{Z}_A$ is determined
by a unitary $U \in B(H)$:
if $\mathbb{Z}_A = \{0, 1, \ldots, |A|-1\}$
$\rho(i) = U^i$
One has $U^{|A|} = 1$, i.e. U has eigenvalues $\in$
$\{1, \omega, \ldots, \omega^{|A|-1}\}$
where $\omega = e^{\frac{2\pi i}{|A|}}$

Let E_j be the corresponding projections
onto eigenvalues
we have $E_i \perp E_j$ $i \neq j$
$$\sum_1^{|A|} E_i = 1$$
$$E_i = \frac{1}{|A|} \sum_{r=0}^{|A|-1} (\omega^{-i} U)^r$$

Let $\tilde{e}_j = \frac{1}{|A|} \left(\sum_{r=0}^{|A|-1} (\omega^{-j} u_1)^r \right.$ in $\mathbb{C}[\mathbb{Z}_A]$

Then $e_j \longmapsto \tilde{e}_j$ is a $*$-isomorphism
between ℓ^∞_A and $C^*(\mathbb{Z}_A)$

The free product
$\ell^\infty_A *_1 \ldots *_1 \ell^\infty_A \simeq C^*(\mathbb{Z}_A * \ldots * \mathbb{Z}_A)$
the free product of groups

If $|A| = 2$
$C^*(\underbrace{\mathbb{Z}_A * \ldots * \mathbb{Z}_A}_{n}) = C^*(F_n)$ the free group of F_n

Thm 1 (Paulsen - Severini - Todorov - Winter)

Let $p = \{p(a,b|x,y)\}$ - synchronous correlation

Then

- $p \in C_{qc}$ iff $\exists$ tracial state $\tau : \mathcal{A}_{X,A} \to \mathbb{C}$ s.t. $p(a,b|x,y) = \tau(e_{xa} e_{yb})$
- $p \in C_q$ iff $\exists$ finite-dim C^*-algebra $\mathcal{A}$ and tracial $\tau : \mathcal{A} \to \mathbb{C}$, $\pi : \mathcal{A}_{X,A} \to \mathcal{A}$ $*$-homomorphism s.t. $p(a,b|x,y) = \tau(\pi(e_{xa} e_{yb}))$
- $p \in C_{loc}$ iff $\mathcal{A}$ can be choosen commutative
- $p \in \overline{C_q}$ iff $\mathcal{A} = \mathcal{R}^\omega$, τ_ω ultrapower of hyperfinite II_1-factor (important for Connes Embedding Problem)

Kim-Paulsen-Schafhauser

A refinement for games:

$\mathcal{G} = (X, X, A, A, \lambda)$ - a synchronous game

closed ideal $I(\mathcal{G})$ in $\mathcal{A}_{X,A}$ generated by $\{e_{xa} e_{yb} : \lambda(x,y,a,b) = 0\}$

Let $A(\mathcal{G}) = \mathcal{A}_{X,A} / I(\mathcal{G})$. Then

Thm 2 (Helton - Meyer - Paulsen - Satarino)

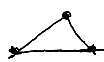
$\mathcal{G}$ has a perfect local strategy iff $\exists$ unital $*$-hom. $A(\mathcal{G}) \to \mathbb{C}$

— " — q-strategy iff — " — $A(\mathcal{G}) \to M_n(\mathbb{C})$

— " — qc-strategy iff — " — $A(\mathcal{G}) \to$ tracial C^*-alg.

Conclusion: looking at the game algebra
$\mathcal{A}(\mathcal{G})$ whether there exist
perfect strategies
in certain class

It may happen that $I(\mathcal{G})$ contains
an identity and then $\mathcal{A}(\mathcal{G})$ collapses
to zero
and hence no unital $*$-homomorphisms
and no perfect strategies
in any class

Exercise: 1. $\mathcal{G}$ = colouring game of K_3
(the full graph on three vertices
) with 2 colours

show that $1 \in I(\mathcal{G})$.

2. $\mathcal{G}$ = colouring game of K_5
with 4 colours
is impossible:

there are no Hilbert space proj.
$\{e_{x,a}\}$ $\quad a \in \{1,2,3,4\}$
$x \in \{1,2,3,4,5\}$

such that $\sum_a e_{xa} = 1 \quad \forall x$

and $e_{x,a} e_{y,a} = 0 \quad x \neq y$

In fact, consider
$q_a = \sum_x e_{x,a}$. $\quad q_a = q_a^*$
$q_a = q_a^2$ $\leftarrow$ check

$\sum_a (1 - q_a) = 4 - \sum_{a,x} e_{xa} = 4 - 5 = -1$
which is impossible.

Idea of the proof
of Thm 1:

Let $p \in C_{qc}^{s}$. Then $p(a,b|x,y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle$
for PVMs $\{E_{x,a}\}$ and $\{F_{y,b}\}$
$[E_{x,a}, F_{y,b}] = 0$.

Let $k = |A| = |B|$

Then $1 = \sum_{a,b=1}^{k} p(a,b|x,y) = \sum_{a=1}^{k} p(a,a|x,x) =$

$$= \sum_a \langle E_{xa} F_{xa} \xi, \xi \rangle =$$

$$= \sum_a \langle F_{x,a} E_{x,a} \xi, \xi \rangle = \sum_a \langle E_{x,a} \xi, F_{x,a} \xi \rangle$$

$$\underset{\text{Cauchy-Schw.}}{\leq} \sum_a \|E_{x,a}\xi\| \, \|F_{x,a}\xi\| \underset{\text{Cauchy-Schw.}}{\leq} \left(\sum_a \|E_{x,a}\xi\|^2\right)^{1/2} \times$$

$$\times \left(\sum_a \|F_{x,a}\xi\|^2\right)^{1/2} =$$

$$= \left(\sum_a \langle E_{x,a}\xi, E_{x,a}\xi \rangle\right)^{1/2} \left(\sum_a \langle F_{x,a}\xi, F_{x,a}\xi \rangle\right)^{1/2}$$

$$= \left(\begin{matrix} E_{x,a} \text{ - proj} \\ F_{x,a} \end{matrix}\right) = \left(\sum_a \langle E_{x,a}\xi, \xi \rangle\right)^{1/2} \left(\sum_a \langle F_{x,a}\xi, \xi \rangle\right)^{1/2}$$

$$= \left(\sum_a E_{x,a} = 1\right) =$$

$$= \|\xi\| \, \|\xi\| = 1$$

Hence we have ~~equality~~ everywhere

equality for Cauchy-Schwartz
implies $E_{xa}\xi = \lambda F_{xa}\xi$
and easy to see that $\lambda = 1$.

Let $\mathcal{A}$ = the C^*-alg. generated by E_{xa}
and let $\tau(a) = \langle a\xi, \xi \rangle$, $a \in \mathcal{A}$.

Then τ is a trace.

In fact

$$\tau(E_{yb}E_{xa}) = \langle E_{yb}E_{xa}\xi, \xi\rangle =$$
$$= \langle E_{yb}F_{xa}\xi, \xi\rangle = \langle F_{xa}E_{yb}\xi, \xi\rangle$$
$$= \langle E_{yb}\xi, F_{xa}\xi\rangle = \langle E_{yb}\xi, E_{xa}\xi\rangle =$$
$$= \langle E_{xa}E_{yb}\xi, \xi\rangle = \tau(E_{xa}E_{yb})$$

we use $E_{xa}\xi = F_{xa}\xi$
and $[E_{xa}, F_{yb}] = 0$

Exercise: complete the proof about trace by showing that

1. $\tau(z E_{xa}) = \tau(E_{xa} z)$ for all words z in E_{yb}
2. use induction on length of words w in E_{xa} to see $\tau(zw) = \tau(wz)$.

For the converse one uses GNS construction for positive functionals on C^*-alg.

and present
$$\tau(e_{xa}e_{yb}) = \langle \pi(e_{xa})\pi(e_{yb})\xi, \xi\rangle$$
where π is a $*$-repr on H of $\mathcal{A}_{X,A}$ and ξ is a cyclic vector in H

Choosing $E_{x,a} = \pi(e_{x,a}) \leftarrow$ PVM

and letting $F_{x,a}(\sum_r W_r\xi) := \sum_r W_r E_{xa}\xi$
where W_r words in $E_{y,b}$

(Obs! $\{\sum_r W_r\xi\}$ are dense in H)

one shows that $\{F_{x,a}\}$ is POVM
$[F_{x,a}, E_{y,b}] = 0$ and
$$\tau(e_{xa}e_{yb}) = \langle E_{xa}F_{yb}\xi, \xi\rangle$$ $\square$

- Using the characterisation of synchronous correlations

Dykema - Paulsen - Prakash showed that $C_q^s \neq \overline{C_q^s}$

and $C_q \neq \overline{C_q}$ ~ strong Tsirelson's conjecture

(first proved by Slofstra using another method)

<u>Thm.</u> (Dykema - Paulsen - Prakash)
The set $C_q^s(n,2)$ is not closed $\forall n \geq 5$.

<u>Cor.</u> $C_q(n,2)$ is not closed $\forall n \geq 5$

The proof is based on the following result about projections

<u>Thm.</u> (Kruglyak - Rabanovich - Samoilenko)
Let $\Sigma_n = \{\alpha \in \mathbb{R}^+ : \exists P_1, \dots, P_n$ - projections $P_1 + \dots + P_n = \alpha I\}$

Then $\Sigma_2 = \{0, 1, 2\}$
$\Sigma_3 = \{0, 1, \frac{3}{2}, 2, 3\}$
$\Sigma_4 =$ a discrete set of rational numbers
$\Sigma_n =$ a discrete part $\cup$
$[\frac{1}{2}(n - \sqrt{n^2 - 4n}), \frac{1}{2}(n + \sqrt{n^2 - 4n})]$

Moreover, $\underbrace{\Sigma_n^{fin}}_{P_i \text{ on f-d. space}} = \Sigma_n \cap \mathbb{Q}$

- Synchronous $p(a,b|x,y) \rightsquigarrow$ traces τ $\rightsquigarrow$ $\tau(e_{xa}e_{yb})$

- If $|A|=2$ we have PVMs $\{\underset{e_{x,0}}{p_x}, \underset{e_{x,1}}{1-p_x}\}$

- Look at $\tau(p_x p_y) = p(0,0|x,y)$

- C_q^s comes from traces on f-d C*-alg. which are $\bigoplus_{k\geq 1} M_{n_k}$

$$p(a,b|x,y) = \tau(\pi(e_{xa})\pi(e_{yb}))$$

$$\pi : \mathcal{A} \to \bigoplus_{k} M_{n_k}$$

$\pi(e_{x0}) = P_x$ - proj. on f-d. space!

- Look at projections P_x s.t. $\sum_{x\in X} P_x = \alpha I$ $\quad |X|=n$

Then $\tau(P_x P_y)$ can be expressed as rational expression of α

consequently for P_x on finite-dim. space H $\tau(P_x P_y)$ is rational

Taking irrational $\alpha \in \Sigma_n$ and approximating by rational in $\Sigma_n \cap \mathbb{Q}$

we get correlation not in C_q but in the closure.