Lecture 2.

Recap: a non-local game
$\mathcal{G} = (X, Y, A, B, \pi, \lambda)$

distribution of questions

the rule function

strategies: - deterministic (f, g), $f: X \to A$, $g: Y \to B$

- probabilistic $\{p(a, b|x, y)\}$

probability to return (a, b) when given (x, y)

C_{loc} = convex comb. of deterministic

C_q : $p(a, b|x, y) = \langle E_{x,a} \otimes F_{y,b} \xi, \xi \rangle$

where $(E_{x,a})_a$ - POVM on f-dim. space H_A

$(F_{y,b})$ - POVM on f-dim space H_B

unit vector $\xi \in H_A \otimes H_B$

C_{qs} : remove finite-dimensionality of H_A and H_B

C_{qc} : $p(a, b|x, y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle$

where $(E_{x,a})_a$, $(F_{y,b})_b$ are POVMs on universal (inf. dim) Hilbert space H

$\xi \in H$, $\|\xi\| = 1$

s.t. $[E_{x,a}, F_{y,b}] = 0$ $\forall x, y, a, b$

We have the chain:

$$C_{loc} \subset C_q \subset C_{qs} \subset \overline{C_q} = \overline{C_{qs}} \subset C_{qc}$$

All of them are no-signalling strategies/correlations

<u>Def.</u> $\{p(a, b|x, y)\}$ is no-signalling if

i.e. there are well-defined marginals:

$p(a|x) = \sum_b p(a,b|x,y) \quad \forall y$ (the same for all y!)

$p(b|y) = \sum_a p(a,b|x,y) \quad \forall x$

It reflects the property that Alice and Bob are not allowed to communicate

Exercise: Show that C_t are no-signalling for $t \in \{loc, q, qc\}$

Notation: C_{ns}

Def. Let $\mathcal{G}$ be a non-local game. We say that $\mathcal{G}$ has a winning strategy in the class C_t, $t \in \{loc, q, qc, ns\}$ if $\exists\, p \in C_t$ s.t. $\underline{\lambda(x,y,a,b) = 0 \Rightarrow p(a,b|x,y) = 0}$

Define the t-value of the game as

$$w_t(\mathcal{G}) = \sup_{p \in C_t} \sum_{xyab} \pi(x,y)\, p(a,b|x,y)\, \lambda(x,y,a,b)$$

Exercise: If C_t - closed that $\mathcal{G}$ has winning strategy in C_t iff $w_t(\mathcal{G}) = 1$

- Difference between strategies and farther examples.

- $C_{loc}(2,2) \neq C_q(2,2)$

2 input 2 output

Back to CHSH game.

$X = Y = A = B = \mathbb{Z}_2 = \{0, 1\}$

win if $xy = a + b \pmod 2$

We have for $\pi(x,y) = \frac{1}{4}$
the winning rate with deterministic strategies is $\frac{3}{4}$

Also $\omega_{loc}(CHSH) = \frac{3}{4}$

quantum strategies?

For $\theta \in [-\pi, \pi]$ let
$v_\theta = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \in \mathbb{C}^2$ and $v_\theta v_\theta^*$ - the projection onto $\mathbb{C}\langle v_\theta \rangle$.

Consider $E_{0,0} = v_0 v_0^*$ $\quad E_{1,0} = v_{\frac{\pi}{4}} v_{\frac{\pi}{4}}^*$

$F_{0,0} = v_{\frac{\pi}{8}} v_{\frac{\pi}{8}}^*$ $\quad F_{1,0} = v_{-\frac{\pi}{8}} v_{-\frac{\pi}{8}}^*$

$E_{0,1} = 1 - E_{0,0}$ $\quad E_{11} = 1 - E_{1,0}$

$F_{0,1} = 1 - F_{0,0}$ $\quad F_{1,1} = 1 - F_{1,0}$

and $\psi = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1)$

$$p(a,b \mid x,y) = \langle E_{x,a} \otimes F_{y,b} \psi, \psi \rangle$$

Check: $\sum \underset{= 1/4}{\pi(x,y)} 1(x,y,a,b)\, p(a,b \mid x,y) =$

$$\frac{1}{4} \sum 1(x,y,a,b)\, p(a,b \mid x,y) = \frac{3}{4} + \frac{\sqrt{2}}{8} > \frac{3}{4}$$

One can do better with quantum strategies!

- The Mermin-Perez magic square game

Fill 3×3 matrix entries with ± 1 in such a way that the product of numbers
in a row $= 1$
in a column $= -1$

1	1	1
1	-1	-1
-1	1	?

Game:

Referee gives (i, j) — i: row to Alice, j: column to Bob

They return: Alice $(a_{i1}, a_{i2}, a_{i3}) \in \{1,-1\}^3$
s.t. $a_{i1} a_{i2} a_{i3} = 1$

Bob $(b_{1j}, b_{2j}, b_{3j}) \in \{1,-1\}^3$
s.t. $b_{1j} b_{2j} b_{3j} = -1$

Rule: win if $a_{ij} = b_{ij}$

No deterministic winning strategy
as product of all entries
multiplied by rows is 1
and by columns is -1.

Exercise: No winning deterministic strategy
$\iff$ no local winning strategy

But $\exists$ winning quantum strategy

Reason $\exists$ matrices X_{ij} with eigenvalues ± 1
s.t. $X_{1j} X_{2j} X_{3j} = -I$ and $X_{i1} X_{i2} X_{i3} = I$

Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{matrix} I \otimes \sigma_z & \sigma_z \otimes I & \sigma_z \otimes \sigma_z \\ \sigma_x \otimes I & I \otimes \sigma_x & \sigma_x \otimes \sigma_x \\ -\sigma_x \otimes \sigma_z & -\sigma_z \otimes \sigma_z & \sigma_y \otimes \sigma_y \end{matrix} \quad = (X_{ij})_{i,j=1}^{3}$$

Observe that operators in each row
and in each column
commute and can
talk on joint eigenspaces

Define $E_{\underset{\text{row}}{1}, (1,1,1)}$ - proj. onto the eigenspace
corresponding to
eigenvalue $(1,1,1)$ for

operators in row 1.

and so on

$F_{j}, (b_{1j}, b_{2j}, b_{3j}) \leftrightarrow$ projection onto eigenspace (b_{1j}, b_{2j}, b_{3j}) of operators in column j $\otimes 2$

Take $\psi = \left(\frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1)\right)$

the 1st and 3d tensor component is for Alice and 2nd and 4th for Bob

One checks that $p(a, b \mid i, j) = \langle E_{i,a} \otimes F_{j,b} \psi, \psi \rangle$

is a winning C_q-strategy

i.e. $p(a, b \mid i, j) = 0$ if

for $a = (a_{i1}, a_{i2}, a_{i3})$

$b = (b_{1j}, b_{2j}, b_{3j})$

$a_{ij} \neq b_{ij}$

- C^*-algebras

C^*-algebra A is a Banach algebra with involution $*$

$a \mapsto a^*$

and norm s.t.

$\|a^* a\| = \|a\|^2, \ a \in A$

A is unital if there is a unit $e \in A$

$(ea = ae = a \quad \forall a \in A)$

Ex. 1. $A = C(X)$, X - compact, Hausdorff

involution $a \mapsto \bar{a}$

$\|a\| = \sup_{x \in X} |a(x)|$

2. $A = M_n(\mathbb{C}) = B(\mathbb{C}^n, \mathbb{C}^n)$ ← bdd operators on $\mathbb{C}^n$ with operator norm

3. Any C^*-algebra is isometrically $*$-isomorphic to a C^*-subalgebra of $B(H)$ ← bdd op. on Hilbert space H (concrete C^*-algebra)

Recall: $A \in B(H)$ is positive $(A \geq 0)$
iff $\langle A\xi, \xi \rangle \geq 0 \ \forall \xi \in H$
iff $\sigma(A) \subset [0, +\infty)$
iff $A = B^* B$ for some $B \in B(H)$

The last two condition can be used to define positivity in abstract C^*-algebra

Let A, B be unital C^*-algebras

<u>Def.</u> A linear map $\varphi : A \to B$ is called positive if $\varphi(a)$ - positive whenever a is positive

φ is called completely positive if $\forall n \in \mathbb{N}$, $\varphi^{(n)} : M_n(A) \to M_n(B)$

$$\varphi^{(n)}((a_{ij})) := (\varphi(a_{ij}))$$

is positive.

Remark: Not all positive are completely positive!

Let $\varphi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$
$a \mapsto a^T$ ← transpose

It is positive, but $\varphi^{(2)}$ is not:

$$X = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \in M_2(M_2(\mathbb{C})) \text{ is positive}$$

$$\text{but } \varphi^{(2)}(X) = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \text{ is not positive}$$

However, any $\varphi : A \to B$, where either A or B are commutative, is completely positive when it is positive

Exercise: Let $\pi: A \to B$ is a $*$-homomorphism show that π is completely positive

(Use the fact that positive are of the form $b^* b$)

Next thm shows that completely positive are not far from $*$-homomorphism

Thm (Stinespring)
Let $\varphi: A \to B(H)$ be completely positive.
Then $\exists$ a Hilbert space K
a $*$-representation $\pi: A \to B(K)$
$V: H \to K$ - bdd operator s.t.

$$\varphi(a) = V^* \pi(a) V, \quad a \in A.$$

Moreover, if φ is unital (i.e. $\varphi(e) = I$)
then V is an isometry $V^* V = I$

<u>Corollaries</u> 1. $\{A_i\}_{i=1}^n$ - POVM on H. space H
Consider $\ell_n^\infty = (\mathbb{C}^n, \|\cdot\|_\infty)$
It is generated by projections $e_i = (0, \dots, \underset{i}{1}, 0 \dots 0)$

Let $\Phi: \ell_n^\infty \to B(H)$
$e_i \mapsto A_i$

Then Φ is positive and as ℓ_n^∞ is commutative, Φ is completely positive

By Stinespring's thm
$$A_i = V^* \pi(e_i) V$$

As π - $*$-homomorphism
$\pi(e_i) =: P_i$ is a projection
and as $\sum_1^n A_i = 1$, $\sum_1^n P_i = 1$

$\{A_i\}$ - POVM $\rightsquigarrow$ $\{P_i\}$ - PVM
(projection-valued measure)

Using for example, Boca's thm one can show that having a family of POVMs $\{E_{x,a}\}_{a\in A}$, $x\in X$ one can find V and PVMs $\{P_{x,a}\}_{a\in A}$ such that

$$E_{x,a} = V^* P_{x,a} V, \ \forall x\in X, a\in A.$$

In particular if $p\in C_q$ is

$$p(a,b|x,y) = \langle E_{x,a}\otimes F_{b,y}\xi, \xi\rangle$$

for some POVMs $\{E_{x,a}\}_{a\in A}$ and $\{F_{y,b}\}_{b\in B}$

we get PVMs $\{P_{x,a}\}_{a\in A}$
$\{Q_{y,b}\}_{b\in B}$

s.t. $E_{x,a} = V^* P_{x,a} V$
$F_{y,b} = W^* Q_{y,b} W$

and $p(a,b|x,y) = \langle P_{x,a}\otimes Q_{y,b}(V\otimes W)\xi, (V\otimes W)\xi\rangle$

i.e. $p\in C_q$ can be defined with PVMs instead of POVMs.

The same is true for C_{qc}.

Idea of the proof of Stinespring's thm:

1. Take $A\otimes H$ as vector space and sesquilinear form

$$[\sum a_i\otimes\xi_i, \sum a_i\otimes\xi_i] = \sum_{i,j}\langle\varphi(a_j^* a_i)\xi_i, \xi_j\rangle$$

2. φ - completely positive $\Rightarrow$

$[\cdot, \cdot]$ is positive definite

3. Quotient out elements $\sum a_i \otimes \xi_i$ s.t.

$[\sum a_i \otimes \xi_i, \sum a_i \otimes \xi_i] = 0$

and get an inner product on it. Write elements $\widehat{\sum a_i \otimes \xi_i}$

4. Let K = the completion w.r.t. the induced norm

5. Define

$\pi : A \to B(K)$

$\pi(a) (\widehat{\sum a_i \otimes \xi_i}) := \widehat{\sum a a_i \otimes \xi_i}$

and show that it is well-defined and bounded

and hence can be extended to the whole K.

π is also a $*$-homomorphism.

6. Define $V : H \to K$

$V\xi := \widehat{1 \otimes \xi}$ (in unital case)

7. Check $\varphi(a) = V^* \pi(a) V$, $a \in A$.

Remark: If $\varphi : A \to \mathbb{C}$ - positive and hence completely positive

we get $V : \mathbb{C} \to K \leadsto$ a vector ξ in K

and

$\varphi(a) = \underline{V^* \pi(a) V} = \langle \pi(a)\xi, \xi \rangle$

May assume $\overline{\pi(A)\xi} = K$ and get a GNS (Gelfand-Naimark-Segal) construction.