Lecture 1.

Non-local games, quantum correlations and operator algebras.

1 Two players game: Alice and Bob play against a Referee

X, Y - finite sets of questions for Alice and Bob

A, B - finite sets of answers — " —

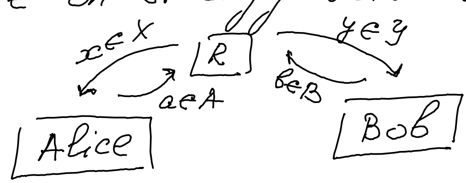
$\lambda : X \times Y \times A \times B \to \{0, 1\}$ - rule function

$\lambda(x, y, a, b) = 1$ win

loose otherwise

- Alice and Bob play cooperatively against Referee
- aware of the rule
- NOT allowed to communicate during the game
- may agree on strategy before the game

One round:



Multiple round: repeat the game

win the game = win each round

Ex. Graph colouring game

G - a simple graph

$X = V(G)$ - set of verticies

$E \subset X \times X$ - edges

- no loops: $(x, x) \notin E$
- undirected: $(x, y) \in E \Rightarrow (y, x) \in E$

write $x \sim y$ if $(x,y) \in E$

The game $X = Y = V(G)$

$A = B$ - set of colours

win: $x \sim y \Rightarrow a \neq b$

$x = x \Rightarrow a = b$

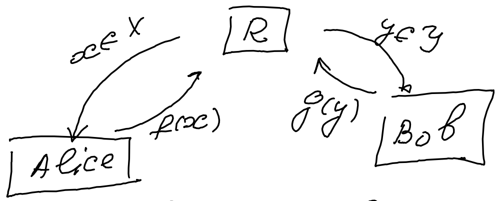
i.e. $\lambda(x, y, a, b) = 0$ if $x \sim y$ and $a = b$

$\lambda(x, x, a, b) = 0$ if $a \neq b$

otherwise 1

2. <u>Strategies.</u>

- Deterministic : pair of funct. $f : X \to A$, $g : Y \to B$



winning if $\lambda(x, y, f(x), g(y)) = 1 \quad \forall x, y$

<u>Exercise.</u> $\exists$ winning determ. strategy for colouring game $\iff$ $\exists$ actual colouring of the graph

winning deterministic NOT always exists

Can talk about probability of winning

Let $\{\pi(x,y)\}_{(x,y)}$ - probability distribution for questions which Alice and Bob know in advance.

Ex. The CHSH-game (Clauser-Horne-Shimony-Holt)

$X = Y = A = B = \mathbb{Z}_2 = \{0, 1\}$

to win need $xy = a + b \pmod 2$

Choose $\pi(x, y) = \frac{1}{4}$ - the uniform distribution $\forall (x, y)$

if (f, g) is such that $f(X) = \{0\} = g(Y)$

the probability to win $= \frac{3}{4}$

It is the best possible with uniform π $\leftarrow$ show

Consider memoryless games,
Then Alice and Bob can answer differently to the same question in different rounds

$\leadsto$ probabilistic strategies

$\{p(a,b|x,y)\}_{a,b}$ $\leftarrow$ prob. distribution $\forall (x,y)$

probability to return (a,b) to (x,y)

For deterministic strategy (f,g)
$$p(a,b|x,y) = \begin{cases} 1, & f(x)=a,\ g(y)=b \\ 0, & \text{otherwise} \end{cases}$$

Local strategies : $\{f_i, g_i\}_{i=1}^n$ - family of determ. strategies

Alice and Bob choose which to use according to prob. distribution $\{\lambda_i\}_{i=1}^n$

Then $p(a,b|x,y) = \sum\limits_{\substack{i: f_i(x)=a \\ g_i(y)=b}} \lambda_i$ $\leftarrow$ convex combination of determ. strategies

More general : (Ω, μ) - probability space which Alice and Bob share

$f_x : \Omega \to A$ for Alice

$g_y : \Omega \to B$ for Bob

Input $(x,y) \leadsto$ evaluate $(f_x(\omega), g_y(\omega))$
(ω - a hidden variable)

$(*)$ $p(a,b|x,y) = \mu(\{\omega : f_x(\omega) = a,\ g_y(\omega) = b\})$

C_{loc} = "all probabilities" of type $(*)$
local correlations.

Known that for CHSH any such can do no better than $\frac{3}{4}$ with $\pi(x,y)=\frac{1}{4}$.

Other physically realizable random strategies

- quantum strategies

Alice and Bob have separated labs and share a possibly entangled state and make measurements for each input

Alice has a state space H_A
$\forall x \in X$ has experiment with $|A|$ outputs

Quantum measurements are described by POVMs (positive operator valued measures)
i.e. $(E_{x,a})_{a\in A}$ with $E_{x,a} \geq 0$ $\sum_a E_{x,a} = I$ for each x.

If the system in a state $\varphi \in H_A$, $\|\varphi\|=1$
the probability to observe a when get x is
$p(a|x) = \langle E_{x,a}\varphi, \varphi\rangle$

Bob has a state space H_B
measuring operators : $(F_{y,b})_{b\in B}$ - POVMs $\forall y \in Y$.

Combined lab: $H_A \otimes H_B$
$\xi \in H_A \otimes H_B$ (possibly entangled i.e. not elementary tensor $\xi_1 \otimes \xi_2$)
that Alice and Bob have access to

Alice makes measurement on its part of ξ
Bob —//—

The probability to return (a,b) when get (x,y) is

$$p(a,b|x,y) = \langle E_{x,a} \otimes F_{y,b} \xi, \xi \rangle$$

The set of all such $\{p(a,b|x,y)\}_{a,b,x,y}$ with finite-dim H_A and H_B are quantum strategies (correlations)

Notation: $\underline{\underline{C_q}}$

If H_A and H_B are allowed to be infinite dimensional get the class $\underline{\underline{C_{qs}}}$

- <u>quantum</u> <u>commuting</u> <u>strategies</u>

$\exists$ universal state space H (can be infinite dimensional)

$\{E_{x,a}\}$ - POVMs for Alice on H

$\{F_{y,b}\}$ - POVMs for Bob on H

Assume $E_{x,a} F_{y,b} = F_{y,b} E_{x,a}$ $\forall x,y,a,b$

and $p(a,b|x,y) = \langle E_{x,a} F_{y,b} \xi, \xi \rangle$ $\xi \in H$, $\|\xi\| = 1$

All such p $\rightsquigarrow$ quantum commuting correlations (strategies)

Notation: $\underline{C_{qc}}$

Facts: C_{loc} - convex, convex combinations of deterministic strategies

Σ extreme points

closed

Exercise

Hint: $p(a,b|x,y) = \int \chi_{\Omega_{a,x}} \chi_{\Omega'_{b,y}} \, d\mu(\omega)$

where $\Omega_{a,x} = \{\omega : f_x(\omega) = a\}$
$\Omega_{b,y} = \{\omega : g_y(\omega) = b\}$

χ_S - the indicator function of $S \subset \Omega$

In $H = L^2(\Omega, d\mu)$

consider $\xi(x) = 1$

and $(E_{x,a}\eta)(\omega) = \chi_{\Omega_{a,x}}(\omega)\eta(\omega)$

$(F_{y,b}\eta)(\omega) = \chi_{\Omega'_{b,y}}(\omega)\eta(\omega)$

$(E_{x,a})_a$, $(F_{y,b})_b$ - POVMs (even all projections!)

Moreover, all projections commute

$$p(a,b|x,y) = \langle E_{x,a}F_{y,b}\xi, \xi\rangle$$

write $\xi = \sum \lambda_i \xi_i$

$\xi_i \in$ joint eigenspace for all $\{E_{x,a}, F_{y,b}\}$

Show $p(a,b|x,y) = (\lambda_i)$ - combination of some determ. strategies.

- C_q - convex

$$\sum_1^k \lambda_i \langle E^i_{x,a} \otimes F^i_{y,b}\xi_i, \xi_i\rangle \ \ominus$$

Let $\xi = (\sqrt{\lambda_i}\xi_i)_{i=1}^k \in (\mathbb{C}^k \otimes H_A) \otimes (\mathbb{C}^k \otimes H_B)$

(Obs! by enlarging H_A and H_B we may assume all $E^i_{x,a}$ act on the same H_A

$F^i_{y,b}$ —"— H_B)

we think of ξ as vector supported on (i,i)-entries in $(\mathbb{C}^k \otimes H_A) \otimes (\mathbb{C}^k \otimes H_B)$

Let $E_{x,a} = \bigoplus_i E^i_{x,a}$, $F_{y,b} = \bigoplus_i F^i_{y,b}$

$\ominus$ $\langle E_{x,a} \otimes F_{y,b}\xi, \xi\rangle$

- $C_{loc} \subset C_q$ $\rightsquigarrow$ show
  use that C_{loc} - convex hull of determ. strategies
  and deterministic $\rightsquigarrow$ $E_{x,f(x)} = 1$ $F_{y,g(y)} = 1$
  0 otherwise
  $\xi = 1 \in \mathbb{C} \otimes \mathbb{C}$

- $C_q \subset C_{qc}$
  $H = H_A \otimes H_B$, $E_{x,a} \otimes 1$ commute with $1 \otimes F_{y,b}$

- C_{qc} - closed (non-trivial)
  and hence $\overline{C_q} \subset C_{qc}$

We have the chain:

$$C_{loc} \subset C_q \subset C_{qs} \subset \overline{C_q} = \overline{C_{qs}} \subset C_{qc}$$

Tsirelson's problem: $C_{qc} = \overline{C_q}$?

equivalent to Connes Embedding Problem have positive answer (in operator algebra)

due to results of Junge et al and Ozawa

Tsirelson's problem has negative solution due to work by Ji, Natarajan, Vidick, Wright, Yuen
$MIP^* = RE$

Hence also Connes emb. problem.

$C_q = \overline{C_q}$ - strong Tsirelson's conjecture

Not true: Slofstra 2019

another proof by

Dykema - Paulsen - Prakash

2019

using methods from

operator algebras

$C_{loc} \neq C_q$ - Bell inequality, 1964.