Lecture 3

Divergence free Kronecker webs in 4D.
Relation with heavenly PDEs and vacuum self-dual Einstein metrics in neutral signature.

Mason-Newman formalism (1989): Thm. Let ω be a volume form on M^4, $X_1(\lambda)=V_1+\lambda V_2$, $X_2(\lambda)=V_3+\lambda V_4$ parameter depending vector fields on M such that

(1) $[X_1(\lambda),X_2(\lambda)]=0$ $(\Leftrightarrow [V_1,V_3]=0, [V_2,V_4]=0, [V_1,V_4]+[V_2,V_3]=0)$

(2) $\mathrm{div}_\omega X_i(\lambda)=0$ $(\Leftrightarrow \mathrm{div}_\omega V_i=0)$.

Then $g=\varkappa(V^1\odot V^4-V^2\odot V^3)$, where $\varkappa=\omega(V_1,V_2,V_3,V_4)$, (V^i) is the coframe dual to the frame (V_i), is self-dual (of signature $(2,2)$) and satisfies vacuum Einstein equations. This correspondence is one-to-one.

Digression on self-duality etc.: $(M^4,g) \rightsquigarrow \underset{\text{Hodge star}}{*}:\Lambda^2T^*M\to\Lambda^2T^*M$

Signatures $(++++),(++--)$ $\Rightarrow *^2=1$ $\quad \Lambda^2T^*M=\Lambda^2_+T^*M\oplus\Lambda^2_-T^*M$

Signature $(+---)$ $\Rightarrow *^2=-1$ (no self-duality)

$*^2=1$ case: $R_{abcd}=R_{[ab][cd]} \Rightarrow \mathcal{R}:\Lambda^2T^*M\to\Lambda^2T^*M$

$$\mathcal{R}=\begin{bmatrix} C_+-\frac{R}{12} & \phi \\ \phi & C_--\frac{R}{12}\end{bmatrix}$$

ϕ - traceless Ricci tensor

$C_\pm$ - (anti) self-dual Weyl tensor

R - Ricci scalar

The metric is selfdual vacuum Einstein if

$$\mathcal{R}=\begin{bmatrix} C_+ & 0 \\ 0 & 0\end{bmatrix}.$$

Plebański heavenly equations (1975):

I Plebański equation $\quad f_{13}f_{24}-f_{14}f_{23}=1 \quad \left(f_{ij}=\frac{\partial^2 f}{\partial x_i\partial x_j}\right)$

$$X_1(\lambda)=-\partial_3+\lambda(f_{13}\partial_2-f_{23}\partial_1)$$
$$X_2(\lambda)=-\partial_4+\lambda(f_{14}\partial_2-f_{24}\partial_1)$$

$$\omega = dx_1\wedge dx_2\wedge dx_3\wedge dx_4$$
$$g = 2(f_{13}dx_1\wedge dx_3 + f_{14}dx_1\wedge dx_4 + f_{23}dx_2\wedge dx_3 + f_{24}dx_2\wedge dx_4)$$

Other equations leading to Mason-Newman vectorfields:

Plebański II equation (1975) $f_{13}+f_{24}+f_{11}f_{22}-f_{12}^2=0$

Husain-Park equation (1992-94) $f_{34}+f_{23}f_{14}-f_{13}f_{24}=0$

Schief or general heavenly equation (1996)
$$(\lambda_1-\lambda_2)(\lambda_3-\lambda_4)f_{12}f_{34}+(\lambda_1-\lambda_3)(\lambda_4-\lambda_2)f_{13}f_{24}$$
$$+(\lambda_1-\lambda_4)(\lambda_2-\lambda_3)f_{14}f_{23}=0$$

Interpretation from the point of view of Kronecker webs:

Mason-Newman vector fields $X_1(\lambda), X_2(\lambda)$ generate a Kronecker web $\{\mathcal{F}_\lambda\}$ of codim 2 on M^4 (two Kronecker blocks of the type $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$) with the additional condition $\operatorname{div}_\omega X_i(\lambda)=0$.
Call such Kronecker webs divergence free.

Remarks 0. It is easy to show that a Kronecker web $\{\mathcal{F}_\lambda\}$ of codim 2 on M^4 is divergence free $\iff$ $T\mathcal{F}_\lambda=\ker\beta_\lambda$,

Gindikin 1982 where, $\beta^\lambda=\beta_0+\lambda\beta_1+\lambda^2\beta_2$ is a 2-form such that $\beta^\lambda\wedge\beta^\lambda=0, d\beta^\lambda=0$.

1. Fixing any $\lambda_1,\lambda_2,\lambda_3$ pairwise distinct one gets a classical 3-web $\{\mathcal{F}_{\lambda_1},\mathcal{F}_{\lambda_2},\mathcal{F}_{\lambda_3}\}$.
2. One can calculate Chern connection for this 3-web, it will be torsionless. Moreover the divergence free condition implies that the Ricci tensor of this connection vanishes.
3. One can choose the parameters $\lambda_1,\lambda_2,\lambda_3$ in such a way that the Chern connection will coincide with the Levi-Civita connection of the corresponding metric g.

Konopelchenko-Schief-Szereszewski observations (2021):

Let $X_1(\lambda), X_2(\lambda)$ be a Mason-Newman vector fields

1. Fix four pairwise distinct parameters $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and use common eigenfunctions φ_i for the vector fields $X_1(\lambda_i), X_2(\lambda_i)$ as coordinates. Then the PNO corresponding to the Kronecker web $\{\mathcal{F}_\lambda\}$ is given by

$N: T\mathcal{F}_\infty \to TM$, $N = \tilde{N}\big|_{T\mathcal{F}_\infty}$, where $\tilde{N}: TM \to TM$ is the Nijenhuis operator $\frac{\partial}{\partial \varphi_i} \mapsto \lambda_i \frac{\partial}{\partial \varphi_i}$ (no summation).

2. There exists a "potential", i.e. a function f such that it satisfies Schief equation and the initial Mason-Newman vector fields are given by

$$X_1(\lambda) = \frac{f_{24}(\lambda_2-\lambda_4)}{f_{12}(\lambda_1-\lambda_2)}(\lambda_1-\lambda)\partial_1 - \frac{f_{14}(\lambda_1-\lambda_4)}{f_{12}(\lambda_1-\lambda_2)}(\lambda_2-\lambda)\partial_2 + (\lambda_4-\lambda)\partial_4, \quad X_2(\lambda) = \frac{f_{23}(\lambda_2-\lambda_3)}{f_{12}(\lambda_1-\lambda_2)}(\lambda_1-\lambda)\partial_1 - \frac{f_{13}(\lambda_1-\lambda_3)}{f_{12}(\lambda_1-\lambda_2)}(\lambda_2-\lambda)\partial_2 + (\lambda_3-\lambda)\partial_3$$

and ω is given by $\omega = f_{12}\, d\varphi_1 \wedge d\varphi_2 \wedge d\varphi_3 \wedge d\varphi_4$

3. If a function f satisfies dispersionless Hirota system of PDEs then it satisfies the Schief equation.

Veronese webs in 4D $\longleftrightarrow$ divergence free Kronecker webs:

$\{V_\lambda\}$, $TV_\lambda = \ker \alpha^\lambda$

$$\alpha^\lambda = \alpha_0 + \lambda\alpha_1 + \lambda^2\alpha_2 + \lambda^3\alpha_3$$

or in the Zakharevich approach

$$\alpha^\lambda = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)\cdot \underbrace{\left(\frac{f_1 dx_1}{\lambda-\lambda_1} + \frac{f_2 dx_2}{\lambda-\lambda_2} + \frac{f_3 dx_3}{\lambda-\lambda_3} + \frac{f_4 dx_4}{\lambda-\lambda_4}\right)}_{((\tilde{N}-\lambda I)^{-1})^t df}$$

$\{\mathcal{F}_\lambda\}$, $T\mathcal{F}_\lambda = \ker \beta^\lambda$

$$\beta^\lambda = \beta_0 + \lambda\beta_1 + \lambda^2\beta_2$$

in the K-S-S approach

$$\beta^\lambda = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)(\lambda-\lambda_4)\cdot \underbrace{\sum_{i<j} \frac{f_{ij}\, dx_i \wedge dx_j}{(\lambda-\lambda_i)(\lambda-\lambda_j)}}_{d((\tilde{N}-\lambda I)^{-1})^t df}$$

$\tilde{N}: T\mathbb{R}^4 \to T\mathbb{R}^4 \quad \frac{\partial}{\partial x_i} \mapsto \lambda_i \frac{\partial}{\partial x_i}$

(*) $\alpha^\lambda \wedge d\alpha^\lambda = 0 \iff$ Hirota system $\overset{d(*)}{\Longrightarrow}$ $\beta^\lambda \wedge \beta^\lambda = 0 \iff$ Schief eq.

$$\beta^\lambda = d\alpha^\lambda$$
$$(d\alpha_3 = 0 \text{ since } \alpha_3 = df)$$

Moreover $\alpha^\lambda \wedge \beta^\lambda = 0$ implies that

$$TV_\lambda \supset T\mathcal{F}_\lambda$$

<u>Remark</u> In fact from a given Veronese web we obtain a 1-parametric family of divergence free Kronecker webs:

$\alpha^\lambda \wedge d\alpha^\lambda = 0 \implies \forall \lambda = \lambda_0 \ \exists$ integrating multiplier g^{λ_0} such that

$$d(g^{\lambda_0} \cdot \alpha^{\lambda_0}) = 0$$

$$\Downarrow$$

$\beta^\lambda = d(g^{\lambda_0} \cdot \alpha^\lambda)$ is of degree 2 in λ.

The forms β^{λ_0} differ essentially for different λ_0: $d(g \cdot \alpha^0) = dg \wedge \alpha^0 + g\, d\alpha^0$

New integrable "heavenly" PDEs (P-Szereszewski, 2022):

Consider different extensions of the PNO $\tilde{N}|_{T\mathcal{F}_\infty} : T\mathcal{F}_\infty \to TM^4$ to globally defined Nijenhuis operators.

Each normal form gives its own integrable PDE.

Open questions:

1. Give a concise characterization of vacuum self dual Einstein metrics coming from Veronese webs
2. Study transformation of canonical connections

on Veronese web to canonical connections
on divergence free Kronecker webs

3. Study particular solutions