Lecture 2

Nijenhuis operators. Veronese webs and Hirota dispersionless systems of PDEs.

A Nijenhuis operator on a manifold M: an endomorfizm $N: TM \to TM$ of the tangent bundle satisfying

(*) $[NX, NY] - N[X,Y]_N = 0$, where $[X,Y]_N := [NX,Y] + [X,NY] - N[X,Y]$, for any vector fields X, Y.

Examples of Nijenhuis operators:

1. If in local coordinates $N = N_i^j \frac{\partial}{\partial x^j} \otimes dx^i$ and $N_i^j = \mathrm{const}$, then N is Nijenhuis

2. Integrable almost complex structure $J: TM \to TM$, $J^2 = -I$ (integrability means that (*) holds)

   the Newlander-Nirenberg theorem: integrability implies existing of coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that $J: \frac{\partial}{\partial x_i} \mapsto \frac{\partial}{\partial y_i}$, $\frac{\partial}{\partial y_i} \mapsto -\frac{\partial}{\partial x_i}$

3. $N: \frac{\partial}{\partial x_i} \mapsto x_i \frac{\partial}{\partial x_i}$ (no summation)

Theorem (many authors) Let ω_1, ω_2 are two symplectic structures on M. Then they are Poisson compatible $\iff$ $\omega_1^{-1} \circ \omega_2 : TM \to TM$ is a Nijenhuis operator

What is an analogue of this theorem for degenerate Poisson structures?

Theorem (Kronecker-Jordan normal form of a pair of linear maps)

Let $F, G: V \to W$ be linear maps. Then there exist a decomposition $V = \bigoplus_i V_i$, $W = \bigoplus_i W_i$, $F = \bigoplus_i F_i$, $G = \bigoplus_i G_i$, $F_i, G_i : V_i \to W_i$ such that in appropriate bases in V_i, W_i the matrices of F_i, G_i are of the following form:

(1) $\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, $\begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & \lambda \end{bmatrix}$ (a Jordan block)

(1) $F_i = \begin{pmatrix} \ddots & \\ 0 & 1 \end{pmatrix}$ $G_i = \begin{bmatrix} \ddots & 1 \\ 0 & \lambda \end{bmatrix}$ ...

(2) $F_i = \underbrace{\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}}_{n+1}\Big\}n$ $G_i = \underbrace{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}}_{n+1}\Big\}n$ (decreasing Kronecker block)

(3) $F_i = \underbrace{\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}}_{n}\Big\}n+1$ $G_i = \underbrace{\begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix}}_{n}\Big\}n+1$ (increasing Kronecker block)

Kronecker bihamiltonian structure: A pair of compatible Poisson structures $\eta_1, \eta_2 : T^*M \to TM$ such there are only Kronecker blocks in the J-K decomposition of the pair $\eta_{1,x}, \eta_{2,x} : T_x^*M \to T_xM$. ($\Leftrightarrow$ $\operatorname{rank}(\lambda_1\eta_1 + \lambda_2\eta_2) = \text{const}$ over $(\lambda_1 : \lambda_2)$)

Remark: Due to the skew symmetry the decomposition contains increasing and decreasing blocks in pairs:

$$\left[\begin{array}{c|c} 0 & \begin{matrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{matrix} \\ \hline \begin{matrix} -1 & & \\ 0 & \ddots & \\ & \ddots & -1 \\ & & 0 \end{matrix} & 0 \end{array}\right] \qquad \left[\begin{array}{c|c} 0 & \begin{matrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{matrix} \\ \hline \begin{matrix} 0 & & \\ -1 & \ddots & \\ & \ddots & 0 \\ & & -1 \end{matrix} & 0 \end{array}\right]$$

Informal explanation of the Gelfand-Zakharevich reduction:

Kronecker bihamiltonian structure

← "almost web"

← common subfoliation the "soul" of bihamiltonian structure (where the action coordinates live) $\Longleftrightarrow$

← web

Kronecker web

increasing + decreasing K.b.

$\downarrow$

increasing K.b.

A partial Nijenhuis operator (Kruglikov, P, Turiel, Zakharevich 2011 - 2016)

Let $\mathcal{F}$ be a foliation on M.

A morfism of bundles $N: T\mathcal{F} \to TM$ is a PNO if

(1) $[X,Y]_N := [NX,Y] + [X,NY] - N[X,Y] \in \Gamma(T\mathcal{F}) \quad \forall X,Y \in \Gamma(T\mathcal{F})$

(2) $[NX,NY] - N[X,Y]_N = 0 \quad \forall X,Y \in \Gamma(T\mathcal{F})$

Remarks: 1. A Nijenhuis operator $N: TM \to TM$ is a PNO with $\mathcal{F} = M$

2. Given a Nijenhuis operator $N: TM \to TM$ and a foliation, the restriction $N|_{T\mathcal{F}}: T\mathcal{F} \to TM$ is a PNO if $N|_{T\mathcal{F}}$ is injective and $N(T\mathcal{F}) \subset TM$ is an integrable distribution

3. Given a PNO $N: T\mathcal{F} \to TM$, the extension of N to a Nijenhuis operator $\tilde{N}: TM \to TM$ (if exists) can be nonunique

4. If $N: T\mathcal{F} \to TM$ is a PNO, then the distribution $(N - \lambda I)(T\mathcal{F}) \subset TM$ is integrable for any $\lambda \in \mathbb{R}$ (here $I: T\mathcal{F} \to TM$ is the canonical inclusion)

Main definition: (Gelfand-Zakharevich, P, Turiel 1995-2001) Let $N: T\mathcal{F} \to TM$ be a PNO such that the pair of operators $N_x, I_x: T_x\mathcal{F} \to T_xM$ is Kronecker for any $x \in M$ (necessarily contains only increasing Kronecker blocks). Then the 1-parametric family of foliations $\{\mathcal{F}_\lambda\}_{\lambda \in \mathbb{P}^1}$ such that $T\mathcal{F}_\lambda = (N - \lambda I)(T\mathcal{F})$, $(\mathcal{F}_\infty := \mathcal{F})$ is called a Kronecker web.

A Veronese web (Gelfand-Zakharevich 1991): A Kronecker web with a sole Kronecker block.

Equivalently: A family of foliations $\{\mathcal{F}_\lambda\}_{\lambda\in\mathbb{P}^1=\mathbb{R}\cup\{\infty\}}$ of codim 1 on M^{n+1} is a Veronese web if locally

$$(T\mathcal{F}_\lambda)^0=\langle \overbrace{\alpha_0+\lambda\alpha_1+\ldots+\lambda^n\alpha_n}^{\alpha^\lambda}\rangle\ ,\quad T\mathcal{F}_\infty:=\langle\alpha_n\rangle^0.$$

linearly independent 1-forms.

$$\mathbb{P}^1\ni\lambda\longmapsto\langle\alpha^\lambda(x)\rangle\in\mathbb{P}T^*_xM$$ – Veronese curve (rational normal curve)

Veronese webs in 3D $\longleftrightarrow$ dispersionless Hirota PDE

Construction of Zakharevich (2001): Let $\{\mathcal{F}_\lambda\}$ be a Veronese web in $\mathbb{R}^3$. Choose $\lambda_1,\lambda_2,\lambda_3$ pairwise distinct. The foliations $\mathcal{F}_{\lambda_1}$, $\mathcal{F}_{\lambda_2}$, $\mathcal{F}_{\lambda_3}$ can be straighten to coordinate hyperplanes $x_1=\text{const}$, $x_2=\text{const}$ and $x_3=\text{const}$. The foliation $\mathcal{F}_\infty$ can be understood as a family of levels of a function f. The formula

$\left(f_i:=\frac{\partial f}{\partial x_i}\right)$

$$\alpha^\lambda=(\lambda-\lambda_2)(\lambda-\lambda_3)f_1dx_1+(\lambda-\lambda_3)(\lambda-\lambda_1)f_2dx_2+(\lambda-\lambda_1)(\lambda-\lambda_2)f_3dx_3$$

gives a unique Veronese curve with $\langle\alpha^{\lambda_i}\rangle^0=\{x_i=\text{const}\}$

$$\langle\alpha^\infty\rangle^0=\{f=\text{const}\}$$

The Frobenius integrability condition $\alpha^\lambda\wedge d\alpha^\lambda=0$ is equivalent to

$$(\ast)\qquad(\lambda_2-\lambda_3)f_1f_{23}+(\lambda_3-\lambda_1)f_2f_{31}+(\lambda_1-\lambda_2)f_3f_{12}=0$$

(dispersionless Hirota equation, Veronese web eq.)

Theorem (Zakharevich 2001): there is a 1-1-correspondence between Veronese webs and solutions to $(*)$

Remarks: 1. PDE $(*)$ is integrable in the sense that it is equivalent to Frobenius integrability condition for the pair of vector fields

$$X_1(\lambda) = f_2(\lambda_1-\lambda)\frac{\partial}{\partial x_1} - f_1(\lambda_2-\lambda)\frac{\partial}{\partial x_2},\quad X_2(\lambda) = f_3(\lambda_2-\lambda)\frac{\partial}{\partial x_2} - f_2(\lambda_3-\lambda)\frac{\partial}{\partial x_3}$$

(Lax pair) (of course $T\mathcal{F}_\lambda = \langle X_1(\lambda), X_2(\lambda)\rangle$)

2. Generalization to higher dimension is straightforward (Veronese webs $\longleftrightarrow$ solutions to a system of PDEs)

Interpretation of the Zakharevich construction from the point of view of Nijenhuis operators: consider a Nijenhuis operator

$$N: \frac{\partial}{\partial x_i} \mapsto \lambda_i \frac{\partial}{\partial x_i} \text{ (no summation)}$$

Then the corresponding PNO is

$$N|_{T\mathcal{F}_\infty} : T\mathcal{F}_\infty \to T\mathbb{R}^3$$

$$\mathcal{F}_\infty = \{f = \text{const}\}$$

The annihilating form can be written as

$$\alpha^\lambda = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)\left((N-\lambda I)^{-1}\right)^t df$$

Taking other normal forms of Nijenhuis operators in 3D gives other integrable PDEs (P-Kruglikov, 2016)

$$\begin{bmatrix} x_1 & & 0 \\ & x_2 & \\ 0 & & x_3 \end{bmatrix}, \quad \begin{bmatrix} \begin{array}{cc} x_2 & 1 \\ & x_2 \end{array} & \\ \hline & x_3 \end{bmatrix}, \quad \begin{bmatrix} x_3 & 0 & 1 \\ 1 & x_3 & -x_2 \\ 0 & 0 & x_3 \end{bmatrix}, \quad \begin{bmatrix} x_1 & -x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & x_3 \end{bmatrix}$$