Kronecker webs and nonlinear PDEs

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PLAN:

Lecture 1. Classical and Kronecker webs. Nijenhuis operators. Application to local geometry of bihamiltonian structures.

Lecture 2. Veronese webs and Hirota dispersionless systems of PDEs.

Lecture 3. Divergence free Kronecker webs in 4D. Relation with heavenly PDEs and vacuum self-dual Einstein metrics in neutral signature.

Lecture 1

1. Short reminder on foliations and distributions

Foliation $\mathcal{F}$ of codim m on a manifold M: a system of submanifolds locally given by equations $x_1 = c_1, \dots, x_m = c_m$, $c_i = \mathrm{const}$, where $(x_1, \dots, x_n)$ is a system of local coordinates on M.

Tangent bundle $T\mathcal{F}$ to a foliation $\mathcal{F}$: the subbundle of the tangent bundle TM formed at a point $p \in M$ by the tangent

space $T_p\mathcal{F} := T\mathcal{F}_p$ ($\mathcal{F}_p$ – leaf of the foliation $\mathcal{F}$ passing through p)

Distribution $\mathcal{D}$ of codim m (corank m) on M: a subbundle $\mathcal{D} \subset TM$ of codim m (a system of subspaces $\mathcal{D}_p \subset T_pM$ locally spanned by $n-m$ linearly independent vector fields $X_1, \dots, X_{n-m}$, or alternatively, locally annihilated by m linearly independent 1-forms $\alpha_1, \dots, \alpha_m$)

Integrable distribution: distribution $\mathcal{D}$ such that there exists a foliation $\mathcal{F}$ with $\mathcal{D} = T\mathcal{F}$

The Frobenius theorem I: a distribution $\mathcal{D}$ is integrable $\Longleftrightarrow$ it is involutive, i.e. $[X_i, X_j] \in \mathrm{Span}(X_1, \dots, X_{n-m})$

The Frobenius theorem II: a distribution $\mathcal{D}$ is integrable $\Longleftrightarrow$ the annihilating 1-forms $\alpha_1, \dots, \alpha_m$ generate a differential ideal, i.e. there exist 1-forms $\beta_{11}, \dots, \beta_{mm}$ such that

$$
\begin{aligned}
d\alpha_1 &= \beta_{11} \wedge \alpha_1 + \dots + \beta_{1m} \wedge \alpha_m \\
&\;\vdots \\
d\alpha_m &= \beta_{m1} \wedge \alpha_1 + \dots + \beta_{mm} \wedge \alpha_m
\end{aligned}
$$

(Codim 1 case: $d\alpha = \beta \wedge \alpha \iff \alpha \wedge d\alpha = 0$)

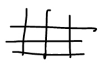
Example of a nonintegrable distribution $\mathcal{D}$ on $\mathbb{R}^3$: $\mathcal{D} = \mathrm{Span}(X_1, X_2)$

$X_1 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$, $X_2 = \frac{\partial}{\partial z}$, $[X_1, X_2] = -\frac{\partial}{\partial y} \notin \mathrm{Span}(X_1, X_2)$

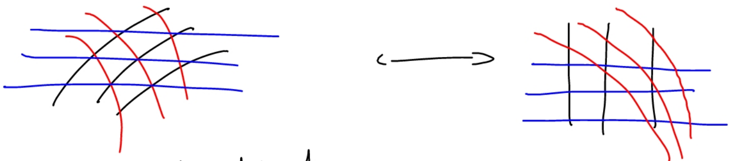
(a contact structure on $\mathbb{R}^3$)

2. Classical webs

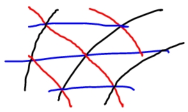
A k-web of codim n on a manifold M: a finite set of foliations $\{\mathcal{F}_1, \dots, \mathcal{F}_k\}$ of codim n in a general position (i.e. the tangent spaces $T_p\mathcal{F}_1, \dots, T_p\mathcal{F}_k$ are in general position at any $p \in M$)

Makes sense if $k > n = \dim M$ (otherwise $\mathcal{F}_i$ can be locally straighten to coordinate planes )

First nontrivial example: a 3-web on a plane



Hexagon invariant:



Generalization I: $(n+1)$-web of codim 1 on M^n

general position condition: any n of n+1 annihilating 1-forms $\alpha_1, \dots, \alpha_{n+1}$ are linearly independent

Generalization II: 3-web of codim n on M^{2n}

Chern connection on 3-webs (1938)

P. Nagy result (2001): Assume a 3-web of codim n is given on M^{2n},

$T\mathcal{F}_1 = \mathrm{Span}(X_1, \dots, X_n)$, $T\mathcal{F}_2 = \mathrm{Span}(Y_1, \dots, Y_n)$, $T\mathcal{F}_3 = \mathrm{Span}(X_1+Y_1, \dots, X_n+Y_n)$,

such that the torsion of the Chern connection is zero.

Then the distribution $\mathrm{Span}(\lambda X_1 + \mu Y_1, \dots, \lambda X_n + \mu Y_n)$ is integrable for any $\lambda, \mu \in \mathbb{R}$, i.e. we have 1-parametric family $\{\mathcal{F}_{(\lambda:\mu)}\}$ of foliations parametrized by $\mathbb{P}^1$ such that $\mathcal{F}_1 = \mathcal{F}_{(1:0)}$, $\mathcal{F}_2 = \mathcal{F}_{(0:1)}$, $\mathcal{F}_3 = \mathcal{F}_{(1:1)}$.

Such a family is an example of a <u>Kronecker web</u> introduced by I. Gelfand and I. Zakharevich (1991-2001)

Poisson structure η on a manifold M: a section η of $\Lambda^2 TM$ satisfying

$[\eta, \eta] = 0$, where $[\,,\,]$ is a Schouten bracket

locally $\eta = \eta^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$, $[\eta, \eta]^{ijk} = \sum_r \eta^{ir} \frac{\partial \eta^{jk}}{\partial x^r}$

i,j,k

Example I: let $\eta^{ij}(x)$ be a nondegenerate matrix and $\omega_{kl}(x)$ the inverse matrix. Then $[\eta,\eta]=0 \iff$
$d\omega = 0$, $\omega = \omega_{kl}\, dx^k \wedge dx^l$ (a symplectic form)

Example II: let $(V, [\,,])$ be a Lie algebra, $e_1, \dots, e_n$ — basis of V
$[e_i, e_j] = c_{ij}^k e_k$. Then $\eta = c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$
is a Poisson structure on V^* ($x_1 = e_1, \dots, x_n = e_n$).
The matrix $c_{ij}^k x_k$ is never nondegenerate

Symplectic (generalized) foliation of a Poisson structure η:
the generalized foliation tangent to the distribution
$D = \mathrm{Span}(X_1, \dots, X_n)$, $X_i = \eta^{ij} \frac{\partial}{\partial x_j}$
(generalized — the vector fields $X_1, \dots, X_n$ may become linearly dependent at some points, i.e. the leaves of the foliation may have nonconstant dimension)
Example of a generalized foliation: trajectories of a vector field with zeroes  $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$

The leaves of the symplectic foliation always are of even dimension and possess a symplectic structure

Examples of symplectic foliations: I — $\mathcal{F} = M$ (one leaf)
II — coadjoint orbits of the Lie algebra on V^*

$so(3)$: $\eta = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$

A bihamiltonian (bi-Poisson) structure on M: a pair (η_1, η_2) of Poisson structures such that $\lambda_1 \eta_1 + \lambda_2 \eta_2$ is a Poisson structure for any $\lambda_1, \lambda_2 \in \mathbb{R}$
(η_1, η_2 are said to be compatible)

The Darboux theorem for a symplectic structure: for a symplectic form ω locally there always exist coordinates (x^i) such that $\omega = dx^1 \wedge dx^2 + \dots + dx^{2n-1} \wedge dx^{2n}$

Similar theorem holds for a Poisson structure of constant rank (rank of the matrix η^{ij} is constant and equal $2n$): $\eta = \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} + \dots + \frac{\partial}{\partial x^{2n-1}} \wedge \frac{\partial}{\partial x^{2n}}$

The Darboux theorem does not hold for pairs of Poisson compatible symplectic forms and compatible Poisson structures of constant rank

Idea of Gelfand and Zakharevich: Let (η_1, η_2) be a pair of compatible Poisson structures such that the rank of $\eta^\lambda = \lambda_1\eta_1 + \lambda_2\eta_2$ is constant in $\lambda = (\lambda_1 : \lambda_2) \neq (0:0)$. Then their symplectic foliations form an "almost web"