SUMMARY OF THE PREVIOUS LECTURES

- WE HAVE DEFINED THE NOTION OF A CONTACT MANIFOLD $(M, \mathcal{C})$
- WE HAVE DEFINED THE NOTION OF A SYMPLECTIC PRINCIPAL $\mathbb{R}^\times$-BUNDLE (P, M, τ, h, ω)
- WE HAVE SHOWN THAT THERE IS ONE-TO-ONE CORRESPONDENCE BETWEEN $(M, \mathcal{C})$ AND (P, M, τ, h, ω) UP TO A DIFFEOMORPHISM

(1) - WE HAVE DEFINED THE LOCAL CONTACT HAMILTONIAN VECTOR FIELD FOR (M, η)

$$H : M \longrightarrow \mathbb{R} \qquad X_H^c : \qquad X_H^c \lrcorner\, \eta = -H \qquad X_H^c \lrcorner\, d\eta = dH - R_\eta(H)\,\eta$$

(2) - WE HAVE DEFINED THE (GLOBAL) CONTACT HAMILTONIAN VECTOR FIELD CORRESPONDING TO A HOMOGENEOUS FUNCTION $\mathcal{H} : P \longrightarrow \mathbb{R}$ AS THE PROJECTION OF THE (PROJECTABLE) SYMPLECTIC HAMILTONIAN VECTOR FIELD $X_{\mathcal{H}}$: $d\mathcal{H} = \omega(X_{\mathcal{H}}, \cdot)$

- WE HAVE ESTABLISHED THE RELATION BETWEEN (1) AND (2) NAMELY IF η IS A LOCAL CONTACT FORM FOR A SECTION $\sigma \in Sec(P)$ THEN

$$\tau_* X_{\mathcal{H}}^c = X_H^c \qquad H : M \longrightarrow \mathbb{R} \qquad H(x) = \mathcal{H}(\sigma(x))$$

EXAMPLES

$$X_H^C = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i\frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}$$

- VISCOSITY FORCE

$M = T^*Q\times\mathbb{R}$ $\qquad$ $\mathcal{C} = \left\langle \frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial z}\right\rangle$ $\qquad$ $\eta = dz - \theta_Q$ $\qquad$ $H(p,z) = H_0(p) - \lambda z$

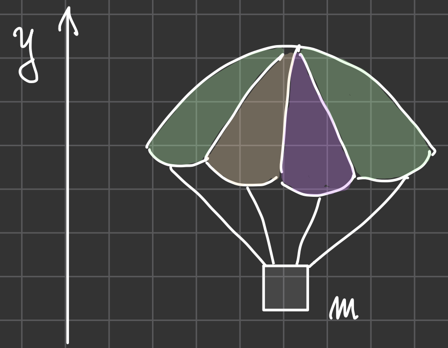
$$X_H^C = \underbrace{\frac{\partial H_0}{\partial p_i}}_{\dot q}\frac{\partial}{\partial q^i} - \underbrace{\left(\frac{\partial H_0}{\partial q^i} - \lambda p_i\right)}_{\dot p}\frac{\partial}{\partial p_i} + \left(p_i\frac{\partial H_0}{\partial p_i} - H_0 + \lambda z\right)\frac{\partial}{\partial z}$$

$$\dot q^i = \frac{\partial H}{\partial p^i}$$

$$\dot p_j = -\frac{\partial H_0}{\partial q^j} + \lambda p_j$$

$$\dot z = \underbrace{p_i\frac{\partial H_0}{\partial p_i} - H_0}_{L_0} + \lambda z$$

- PARACHUTE EQUATION (Xavier Rivas)



$Q = \mathbb{R}$

$M = T^*Q\times\mathbb{R}$

(y,p,z)

$$\mathcal{H}(y,p,z) = \frac{1}{2m}(p-\gamma z)^2 + V(y) \qquad V(y) = \frac{mg}{\gamma}\left(e^{\gamma y} - 1\right)$$

$$\dot y = \frac{1}{m}(p-\gamma z)$$

$$\dot p = -\frac{\partial V}{\partial y} + \frac{\gamma p^2}{m} - \frac{\gamma^2 p z}{m}$$

$$\dot z = \underbrace{\frac{p}{m}(p-\gamma z) - \mathcal{H}}_{L(y,\dot y) = \frac{m}{2}\dot y^2 + \gamma \dot y z - V(y)}$$

$$\ddot y - \gamma m \dot y^2 + g = 0$$

(AN INSTANCE OF HERGLOZ EQUATION)

CONTACT HAMILTON JACOBI THEORY

THE GEOMETRIC CONTENT THAT UNDERLINES MANY VERSIONS OF THE (SYMPLECTIC) HAMILTON-JACOBI THEOREMS IS AS FOLLOWS:

THEOREM: LET (P,ω) BE A SYMPLECTIC MANIFOLD, $\mathcal{H}: P \longrightarrow \mathbb{R}$, $L \subset P$ A LAGRANGIAN SUBMANIFOLD. THEN $X_{\mathcal{H}}$ IS TANGENT TO L IF AND ONLY IF $\mathcal{H}$ IS CONSTANT (LOCALLY) ON L.

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IN HAMILTONIAN MECHANICS WE TAKE: $P = T^*Q$, $\omega = \omega_Q$ $L = dS(Q)$ FOR $S: Q \to \mathbb{R}$

$$H\big(dS(Q)\big) = \text{const}$$

$$H\left(q^i, \frac{\partial S}{\partial q^i}\right) = E$$

MEANS

X_H TANGENT TO $dS(Q)$

PHASE TRAJECTORIES LIE IN $dS(Q)$

PHASE TRAJECTORIES ARE LIFTS OF THE INTEGRAL CURVES OF $\pi_{Q*}\left(X_H|_{dS}\right)$

H-J THEOREM IS VERY SYMPLECTIC

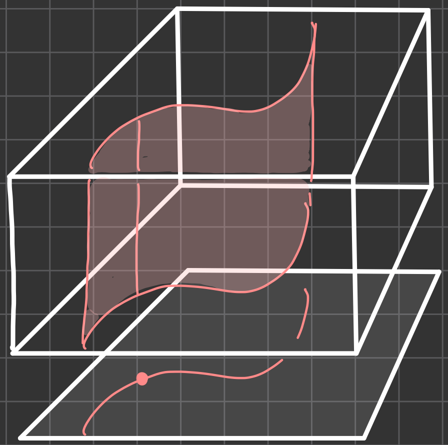
LEGENDRE SUBMANIFOLDS

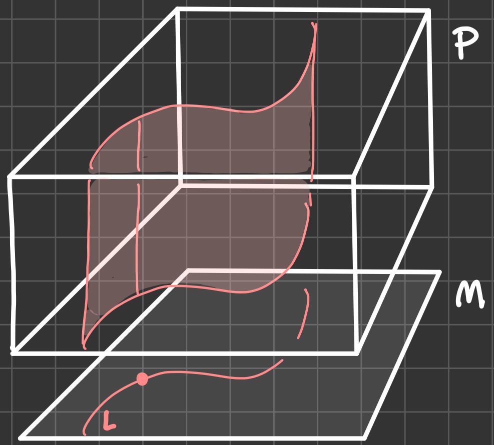
DEFINITION LET (M, C) BE A CONTACT MANIFOLD. A SUBMANIFOLD $N \subset M$ IS CALED ISOTROPIC IF $TN \subset C$. A SUBMANIFOLD $L \subset M$ IS CALED LEGENDRE IF IT IS ISOTROPIC AND OF MAXIMAL DIMENSION

$\dim M = 2n+1 \quad \dim L = n$

IN HOMOGENEOUS SYMPLECTIC LANGUAGE WE HAVE

PROPOSITION LET (P, M, τ, h, ω) DEFINE THE CONTACT STRUCTURE ON M. THEN $L \subset M$ IS A LEGENDRE SUBMANIFOLD IF AND ONLY IF $\tau^{-1}(L)$ IS A LAGRANGIAN SUBMANIFOLD





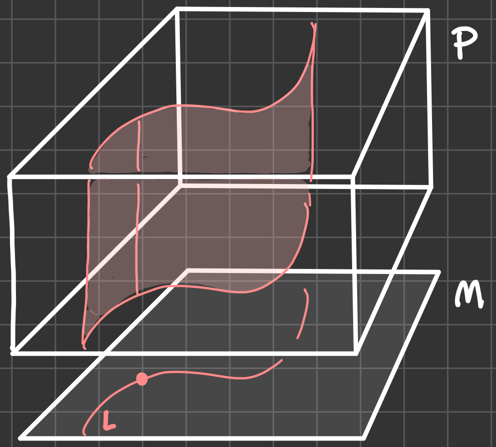
- ON P: $X_{\mathcal{H}}$ IS TANGENT TO $\tau^{-1}(L) \iff \mathcal{H}$ CONSTANT ON L

HOMOGENEOUS

UNION OF FIBERS

CONSTANT MEANS $=0$

- $X_{\mathcal{H}}$ TANGENT TO $\tau^{-1}(L) \implies X^{c}_{\mathcal{H}}$ TANGENT TO L



- ON P: $X_{\mathcal{H}}$ IS TANGENT TO $\tau^{-1}(L)$ $\Longleftrightarrow$ $\mathcal{H}$ CONSTANT ON L

HOMOGENEOUS

UNION OF FIBERS

CONSTANT MEANS $=0$

- $X_{\mathcal{H}}$ TANGENT TO $\tau^{-1}(L)$ $\Longrightarrow$ $X^{c}_{\mathcal{H}}$ TANGENT TO L

THEOREM: LET L BE A LEGENDRE SUBMANIFOLD OF M. A HOMOGENEOUS HAMILTONIAN $\mathcal{H}: P \longrightarrow \mathbb{R}$ IS EQUAL 0 ON $\tau^{-1}(L)$ $\Longleftrightarrow$ $X^{c}_{\mathcal{H}}$ IS TANGENT TO L

NOW WE CAN TRANSLATE IT TO MORE MECHANICAL LANGUAGE

FOR $M = T^{*}Q \times \mathbb{R}$, $M = J^{1}L^{*}$

$M = T^*Q \times \mathbb{R}$

$M = T^*Q \times \mathbb{R}$

$\mathcal{L}_S = \{ (p, z) : \; p = dS(q), \; z = S(q) \} = j^1 S(Q)$

$\mathcal{P} = T^*Q \times \mathbb{R} \times \mathbb{R}$ $\quad \mathcal{H}(q^i, \pi_j, \tau, z) = \tau H\left(q^i, \frac{\pi_j}{\tau}, z\right)$ $\quad \mathcal{H} = 0 \equiv H = 0$

HAMILTON–JACOBI EQUATION : $\quad H\left(q^i, \frac{\partial S}{\partial q^i}, S(q^i)\right) = 0 \quad (*)$

IF S IS A SOLUTION OF THE CONTACT H-J EQUATION (*) THEN SOLUTIONS OF $X^c_{\mathcal{H}}$ WITH INITIAL CONDITIONS ON $\mathcal{L}_S$ CAN BE OBTAINED FROM SOLUTIONS OF $(\pi^1_Q)_* \left(X^c_{\mathcal{H}} \big|_{j^1 S(Q)} \right)$

$\pi^1_Q : T^*Q \times \mathbb{R} \longrightarrow Q$

VECTOR FIELD ON Q

$M = J^1 L^*$

LINE BUNDLE

$$\begin{array}{c} L \\ \tau_0 \downarrow \\ Q \end{array}$$

DUAL

$$\begin{array}{c} L^* \\ \tau_0^* \downarrow \uparrow s \\ Q \end{array}$$

$M = J^1 L^*$ CONTACT MANIFOLD

$j^1 s(Q)$ LEGENDRE SUBMANIFOLD

$(T^* L^\times, J^1 L^*, \tau, d_{T^*} h, \omega_{L^\times})$ SYMPLECTIC $\mathbb{R}^\times$ - BUNDLE

P

$\mathcal{H}_\sigma : T^* L^\times \longrightarrow \mathbb{R}$ HOMOGENEOUS HAMILTONIAN FUNCTION

$\sigma : J^1 L^* \longrightarrow L_P \simeq J^1 L^* \times_Q L^*$ HAMILTONIAN SECTION

HAMILTON JACOBI EQUATION $\quad \sigma(j^1 s) = 0$

AGAIN: SOLUTIONS OF $X^c_{\mathcal{H}}$ WITH INITIAL CONDITIONS ON $j^1 s(Q)$ CAN BE OBTAINED FROM SOLUTIONS OF $\tau^1_*(X^c_{\mathcal{H}}|_{j^1 s(Q)})$

$\tau^1_* : J^1 L^* \longrightarrow Q$

VECTOR FIELD ON Q

FROM PRINCIPAL BUNDLE TO LINE BUNDLE

$\mathbb{R}^\times$ PRINCIPAL BUNDLE

$$\begin{array}{c} P \\ \downarrow \\ M \end{array}$$

$P = L_P^\times$

LINE BUNDLE

$L_P = P \times \mathbb{R} / \mathbb{R}^\times$

$(p, r) \sim (sp, \frac{r}{s})$

DUAL LINE BUNDLE

$L_P^* = P \times \mathbb{R}^* / \mathbb{R}^\times$

$(p, r) \sim (sp, sr)$

HOMOGENEOUS FUNCTIONS ON P CORRESPOND TO SECTIONS OF L_P^*

$M = J^1 L^* \qquad P = T^* L^\times \qquad L_P = ?$

PROPOSITION:

FOR $P = T^* L^\times$ WE GET
$$L_P \simeq J^1 L^* \times_Q L$$
$$L_P^* \simeq J^1 L^* \times_Q L^*$$

THEREFORE THE HAMILTONIAN SECTION CORRESPONDING TO A HOMOGENEOUS HAMILTONIAN FUNCTION ON $T^* L^\times$ IS A MAP

$$\begin{tikzcd} J^1 L^* \arrow[r] \arrow[d] & L \arrow[d] \\ Q \arrow[r, "="] & Q \end{tikzcd}$$

CONTACT LAGRANGIAN MECHANICS

- IN THE LITERATURE:

$$TQ \times \mathbb{R} \ni (v, z) \longmapsto L(v,z) \in \mathbb{R}$$

$$T^*Q \times \mathbb{R} \ni (p, z) \longmapsto H(p,z) \in \mathbb{R}$$

$$H(p,z) = \langle p, v \rangle - L(v,z)$$

LEGENDRE TRANSFORMATION HAPPENS ONLY IN $p \Longleftrightarrow v$
„z" VARIABLE HAS TO BE SEPARATED

- THE DIFFERENTAL CONSEQUENCE OF CONTACT HAMILTONIAN EQUATIONS ARE THE FOLLOWING HERGLOZ EQUATIONS

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) = \frac{\partial L}{\partial q^i} + \frac{\partial L}{\partial z}\frac{\partial L}{\partial \dot{q}^i} \qquad \dot{z} = L(q, \dot{q}, z)$$

- FOR THIS TO WORK WE NEED THE PRODUCT $T^*Q \times M$ STRUCTURE

WHAT WITH $M = J^1 L^*$?

1931 – 2022

CLASSICAL TULCZYJEW TRIPLE

$$\begin{tikzcd}
 & & & & \mathcal{D} \arrow[d, hook] & & & & \\
 & T^*T^*Q \arrow[dl] \arrow[ddr] & & & TT^*Q \arrow[lll, "\beta_Q = \omega_Q^b"'] \arrow[rrr, "\alpha_Q"] \arrow[dl] \arrow[ddr] & & & T^*TQ \arrow[dl] \arrow[ddr] & \\
T^*Q \arrow[ur, bend left, "d\mathcal{H}"] \arrow[rrr, equal] \arrow[ddr] & & & T^*Q \arrow[ur, bend left, "X_{\mathcal{H}}"] \arrow[rrr, equal] \arrow[ddr] & & & T^*Q \arrow[ddr] & & \\
 & & TQ \arrow[rrr, equal] \arrow[dl] & & & TQ \arrow[rrr, equal] \arrow[dl] & & & TQ \arrow[ull, "\lambda"'] \arrow[uul, bend right, "d\mathcal{L}"'] \arrow[dl] \\
 & Q \arrow[rrr, equal] & & & Q \arrow[rrr, equal] & & & Q &
\end{tikzcd}$$

$$\mathcal{D} = X_{\mathcal{H}}(T^*Q) = \beta_Q^{-1}\left(d\mathcal{H}(T^*Q)\right) = \alpha_Q^{-1}\left(d\mathcal{L}(TQ)\right)$$

IN NOT REGULAR CASES $\mathcal{D}$ MAY NOT BE THE IMAGE OF A VECTOR FIELD

$(M,\mathcal{C})$

$$\begin{tikzcd}
T^*P \arrow[d] & & TP \arrow[ll] \arrow[dd] \\
P \arrow[u, bend left, "dH"] \arrow[urr, "X_H"] \arrow[d] & & \\
M \arrow[rr, "X_H^c"] & & TM
\end{tikzcd}$$

$M = J^1L^*$

$$\begin{tikzcd}
T^*T^*L^\times \arrow[d] & & TT^*L^\times \arrow[ll] \arrow[dd] \\
T^*L^\times \arrow[u, bend left, "dH"] \arrow[urr, "X_H"] \arrow[d] & & \\
J^1L^* \arrow[rr, "X_H^c"] & & TJ^1L^*
\end{tikzcd}$$

$$\begin{tikzcd}
& T^*T^*L^\times \arrow[dl] & & TT^*L^\times \arrow[ll, "\beta_{L^\times}"'] \arrow[dl] \arrow[dd] \\
T^*L^\times \arrow[ur, bend left, "dH"] \arrow[rr, equal] \arrow[d] & & T^*L^\times \arrow[ur, bend left, "X_H"] & \\
J^1L^* \arrow[rrr, bend left, "X_H^c"] & & & TJ^1L^* \arrow[lll]
\end{tikzcd}$$

?

$(M, \mathcal{C})$

$$\begin{tikzcd}
T^*P \arrow[d] & & TP \arrow[ll] \arrow[dd] \\
P \arrow[u, bend left, "d\mathcal{H}"] \arrow[urr, "X_{\mathcal{H}}"] \arrow[d] & & \\
M \arrow[rr, "X^c_{\mathcal{H}}"] & & TM
\end{tikzcd}$$

$M = J^1L^*$

$$\begin{tikzcd}
T^*T^*L^\times \arrow[d] & & TT^*L^\times \arrow[ll] \arrow[dd] \\
T^*L^\times \arrow[u, bend left, "d\mathcal{H}"] \arrow[urr, "X_{\mathcal{H}}"] \arrow[d] & & \\
J^1L^* \arrow[rr, "X^c_{\mathcal{H}}"] & & TJ^1L^*
\end{tikzcd}$$

$$\begin{tikzcd}
T^*T^*L^\times \arrow[d] & TT^*L^\times \arrow[l, "\beta_{L^*}"'] \arrow[d] \arrow[dd] \arrow[rr] & & T^*TL^\times \arrow[dl] \arrow[dr] & \\
T^*L^\times \arrow[u, bend left, "d\mathcal{H}"] \arrow[r, equal] \arrow[d] & T^*L^\times \arrow[u, bend left, "X_{\mathcal{H}}"] & T^*L^\times & & TL^\times \arrow[ul, bend right, "dL"'] \\
J^1L^* \arrow[r, bend left, "X^c_{\mathcal{H}}"] & TJ^1L^* \arrow[l] & & &
\end{tikzcd}$$

PHASE SPACE
EQUIPPED WITH
CONTACT STRUCTURE

- WHAT IS THE LAGRANGIAN SPACE
- WHAT IS THE RELEVANT STRUCTURE
- WHAT IS THE LAGRANGIAN ?
- ...

- WHAT IS THE LAGRANGIAN SPACE
- WHAT IS THE RELEVANT STRUCTURE
- WHAT IS THE LAGRANGIAN ?
- ...

$$\begin{array}{c} L^\times \\ \downarrow \\ Q \end{array}$$

$\mathbb{R}^\times$ – PRINCIPAL BUNDLE

μ

$$\begin{array}{c} TL^\times \\ \downarrow \\ AL^\times \end{array} \quad d_T \mu$$

ATIYAH ALGEBROID (AS A BUNDLE OVER Q)

$$AL^\times = TL^\times / \mathbb{R}^\times$$

$$\begin{array}{c} T^*TL^\times , \; d_{T^*} d_T \mu \\ \downarrow \\ J^1 L^*_{TL^\times} \end{array}$$

SYMPLECTIC $\mathbb{R}^\times$ – PRINCIPAL BUNDLE

THE LAGRANGIAN SPACE

$$L^*_{TL^\times} \simeq AL^\times \times_Q L^*$$

$$\begin{tikzcd} \mathcal{L} : AL^\times \arrow[r] \arrow[d] & L^* \arrow[d] \\ Q \arrow[r, "="] & Q \end{tikzcd}$$

$$
\begin{tikzcd}
 & J^1 L^*_{T^*L^\times} \arrow[dl] \arrow[drr, "\beta"] & & AT^*L^\times \arrow[ll, "\beta_0"'] \arrow[rr, "\alpha_0"] \arrow[d] & & J^1 L^*_{TL^\times} \arrow[dll, "\alpha"'] \arrow[dr] & & \\
L & J^1L^* \arrow[l, "\mathcal{H}"'] & & TJ^1L^* & & & AL^\times \arrow[r, "\mathcal{L}"] & L \\
 & & & \mathcal{D} \arrow[u, hook] & & & &
\end{tikzcd}
$$

$$\mathcal{D} = \beta(j^1\mathcal{H}) = \alpha(j^1\mathcal{L})$$

THERE EXISTS THE LAGRANGIAN VERSION OF MECHANICS FOR NONTRIVIAL CONTACT SYSTEMS, BUT LAGRANGIANS ARE SECTIONS OF CERTAIN LINE BUNDLE OR HOMOGENEOUS FUNCTIONS ON $TL^\times$

EXAMPLE WHAT DO WE GET FOR $L = Q\times\mathbb{R}$ AND $H: T^*Q\times\mathbb{R} \longrightarrow \mathbb{R}$ $H(p,z) = H_0(p) - \lambda z$?

$$\mathcal{L}: TQ\times\mathbb{R} \longrightarrow \mathbb{R}$$
$$\mathcal{L}(v,\tau)$$

CONTACT HAMILTONIAN SYSTEM WITH VISCOSITY FORCE IS NOT REGULAR IN THIS SETTING THEREFORE THERE IS NO SINGLE LAGRANGIAN

PARAMETER OF THE FAMILY

$$\mathcal{L}: \overset{v}{TQ}\times\overset{\tau}{\mathbb{R}}\times\overset{z}{\mathbb{R}} \longrightarrow \mathbb{R}$$

$$\mathcal{L}(v,\tau,z) = \mathcal{L}_0(v) + z(\tau-\lambda) \rightsquigarrow$$ HERGLOZ EQUATIONS

THANK YOU!

DZIĘKUJĘ