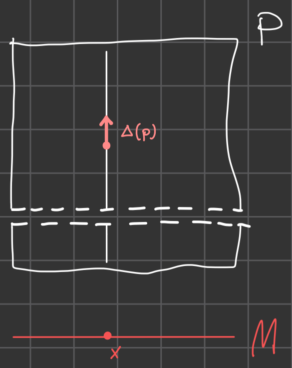
SUMMARY OF THE FIRST LECTURE

- A CONTACT MANIFOLD IS A MANIFOLD M TOGETHER WITH A CORANK 1, MAXIMALLY NONINTEGRABLE DISTRIBUTION $\mathcal{C}$
- LOCALLY $\mathcal{C}$ IS OF THE FORM $\left\langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \frac{\partial}{\partial p_i} \right\rangle$ $i = 1 \dots n$
- LOCALLY $\mathcal{C} = \ker \eta$, $\eta \wedge (d\eta)^n \neq 0$ $\eta = dz - p_i dq^i$
- NATURAL EXAMPLES – CARTAN DISTRIBUTION ON JET SPACES
- A SYMPLECTIC $\mathbb{R}^\times$-PRINCIPAL BUNDLE IS AN $\mathbb{R}^\times$-PRINCIPAL BUNDLE $P \longrightarrow M$ WITH A SYMPLECTIC FORM ω ON P SUCH THAT $h_s{}^* \omega = s\omega$
  (P, M, τ, h, ω)
- EVERY CONTACT STRUCTURE DEFINES A SYMPLECTIC $\mathbb{R}^\times$-PRINCIPAL BUNDLE $P = (C^0)^\times \subset T^*M$, $\tau = \pi_M|_P$, $h_s(\alpha) = s\alpha$ $\omega = \omega_M|_P$

EVERY (P, M, I, h, ω) DEFINES (M, C)

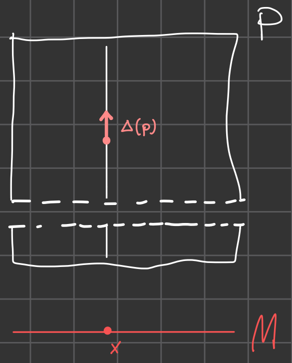


- $\nabla(p) = \frac{d}{dt}\Big|_{t=0} h_{e^t}(p) = \frac{d}{ds}\Big|_{s=1} h_s(p)$

VERTICAL AND INVARIANT

$$\nabla(h_r p) = \frac{d}{ds}\Big|_{s=0} h_s(h_r(p)) = \frac{d}{ds}\Big|_{s=0} h_r(h_s(p)) = Th_r(\nabla(p))$$

EVERY $(P, M, \mathcal{I}, h, \omega)$ DEFINES (M, C)



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HOMOGENEOUS

- $$\theta = i_\nabla \omega$$

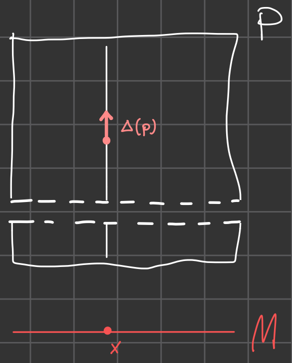
HOMOGENEOUS

INVARIANT

$$d\theta = \omega$$

$$\hookrightarrow h_s^* \omega = s\omega \longrightarrow \mathcal{L}_\nabla \omega = \omega = d i_\nabla \omega + \cancelto{0}{i_\nabla d\omega} = d\theta$$

EVERY (P, M, τ, h, ω) DEFINES (M, C)



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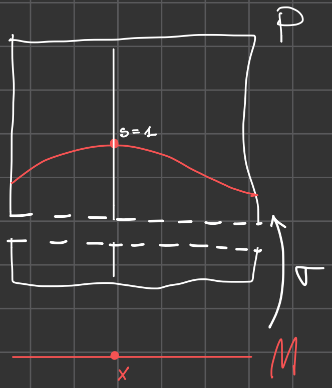


A LOCAL SECTION σ GIVES
A LOCAL TRIVIALISATION
$P \simeq M \times \mathbb{R}^\times$
AND A COORDINATE s

NVARIANT i.E. BASIC

SOME LOCAL FORM ON M

- $$\frac{1}{s}\theta = \tau^* \eta$$

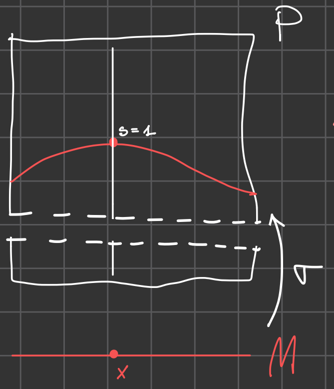
$$\mathbb{R}^\times \times M \ni (s, x) \xmapsto{I_\sigma} s\sigma(x) \in P$$

$$I_\sigma^*(\theta) = s\eta \;, \quad I_\sigma^*(\omega) = d(s\eta) = ds \wedge \eta + s\, d\eta$$

ω SYMPLECTIC ⟶ η CONTACT

- $$C = \ker \eta$$

DOES NOT DEPEND ON
THE CHOICE OF σ
DIFFERENT SECTION LEADS
TO η MULTIPLIED BY A FUNCTION



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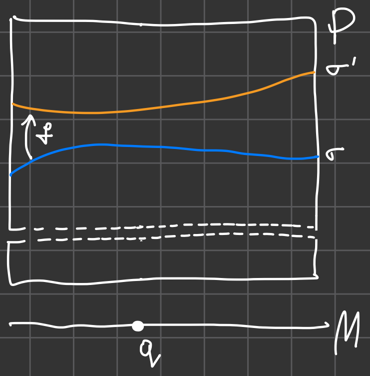

$$\sigma' = f\sigma$$

$$p = s\sigma(q) = s'\sigma'(q) = s' f(q)\sigma(q)$$

$$s = f s'$$

$$\theta = s\tau^{*}\eta = s'\tau^{*}\eta' \qquad f s'\tau^{*}\eta = s'\tau^{*}\eta'$$

$$f\eta = \eta'$$

EXAMPLES

- $T^*Q \longrightarrow \mathbb{P}T^*Q$ PROBABLY THE SIMPLEST WAY TO INTRODUCE CONTACT STRUCTURE ON THE PROJECTIVE COTANGENT BUNDLE

HOMEWORK: FIND A COORDINATE EXPRESSION FOR $\mathcal{C}$ ON $\mathbb{P}T^*Q$

EXAMPLES

- $T^*Q \longrightarrow \mathbb{P}T^*Q$ PROBABLY THE SIMPLEST WAY TO INTRODUCE CONTACT STRUCTURE ON THE PROJECTIVE COTANGENT BUNDLE

- $$\begin{tikzcd} L \arrow[d] \\ M \end{tikzcd}$$ (x^i, τ) LINE BUNDLE

  $$\begin{tikzcd} L^* \arrow[d] \\ M \end{tikzcd}$$ (x^i, z) DUAL LINE BUNDLE

  $$\begin{tikzcd} L^\times = L \setminus \{0_L(m)\} \arrow[d, "\mathbb{R}^\times"] \\ M \end{tikzcd}$$ PRINCIPAL BUNDLE

$T^*L^\times$

IT IS AN $\mathbb{R}^\times$– PRINCIPAL BUNDLE WITH THE LIFTED ACTION $d_{T^*}h_s$

$$(x^i, \tau, p_j, z)$$

$$(Th_s)^*(x^i, \tau, p_j, z) = (x^i, \frac{\tau}{s}, p_j, sz)$$

$$(Th_{\frac{1}{s}})^*(x^i, \tau, p_j, z) = (x^i, s\tau, p_j, \frac{z}{s})$$

$$d_{T^*}h_s(x^i, \tau, p_j, z) = (x^i, s\tau, sp_j, z)$$

$$d_{T^*}h_s = s(Th_{\frac{1}{s}})^*$$

IT IS A SYMPLECTIC MANIFOLD

$$\omega_{L^\times} = dp_i \wedge dx^i + dz \wedge d\tau \qquad (d_{T^*}h_s)^* \omega_{L^\times} = d(sp_i) \wedge dx^i + dz \wedge d(s\tau) = s\,\omega_{L^\times}$$

$(T^*L^\times, ?, ?, d_{T^*}h_s, \omega_{L^\times})$ SYMPLECTIC PRINCIPAL $\mathbb{R}^\times$-BUNDLE

$$T^*L^\times / \mathbb{R}^\times = ?$$

$(T^*L^\times, ?, ?, d_{T*}h_s, \omega_{L^\times})$ SYMPLECTIC PRINCIPAL $\mathbb{R}^\times$-BUNDLE

$$T^*L^\times / \mathbb{R}^\times = ?$$

THERE IS A CORRESPONDENCE BETWEEN

SECTIONS OF

$$\begin{tikzcd} L^* \arrow[d] \\ M \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

HOMOGENEOUS FUNCTIONS

$$f : L^\times \longrightarrow \mathbb{R}$$

$$f_\sigma(\ell) = \langle \sigma(x), \ell \rangle$$

JETS OF SECTIONS

$$j^1\sigma_1(x) = j^1\sigma_2(x)$$

$$\sigma_1(x) = \lambda\sigma_2(x)$$

$$\lambda(x) = 1 \quad d\lambda(x) = 0$$

DIFFERENTIALS OF FUNCTIONS

$$f\sigma_1 = \lambda f\sigma_2$$

$$df_{\sigma_1}(\ell) = d(\lambda f_{\sigma_2}) = \overset{1}{\overset{\shortparallel}{\lambda(x)}} df_{\sigma_2} + f_{\sigma_2}\cancel{d\lambda(x)}^{\,0}$$

AT ℓ OVER x $\quad df_{\sigma_1} = df_{\sigma_2}$

MOREOVER:

$$d_{T*}h_s(df_\sigma(\ell)) = df_\sigma(h_s(\ell))$$

EVERY COVECTOR IS A DIFFERENTIAL OF
SOME HOMOGENEOUS FUNCTION

$$\alpha_i dx^i + \alpha d\tau = df(x_0^i, \tau_0) \qquad f(x^i, \tau) = \tau\left[\frac{\alpha_i}{\tau_0}(x^i - x_0^i) + \alpha\right]$$

$$T^*L^\times / \mathbb{R}^\times = J^1L^*$$

THE CANONICAL CONTACT SRUCTURE OF J^1L^*
CORRESPONDS TO THE $\mathbb{R}^\times$-PRINCIPAL SYMPLECTIC
BUNDLE

$$(T^*L^\times, J^1L^*, \tau, d_{T*}h, \omega_{L^\times})$$

APPLICATIONS:

- CONTACT HAMILTONIAN SYSTEMS
- CONTACT HAMILTON-JACOBI THEOREMS
- CONTACT LAGRANGIAN MECHANICS
- CONTACT REDUCTIONS

⋮

CONTACT HAMILTONIAN VECTOR FIELDS

IN THE LITERATURE, CONTACT HAMILTONIAN DYNAMICS IS FORMULATED IN TERMS OF CONTACT FORM:

(M, η) $\quad R_\eta$ - REEB VECTOR FIELD, $\quad H : M \longrightarrow \mathbb{R}$ HAMILTONIAN

$R_\eta \lrcorner\, \eta = 1$

$R_\eta \lrcorner\, d\eta = 0$

X_H^c : $\quad X_H^c \lrcorner\, \eta = -H \qquad X_H^c \lrcorner\, d\eta = dH - R_\eta(H)\eta$

CONTACT HAMILTONIAN VE

IN THE LITERATURE, CONTA FORMULATED

IN TERMS OF CONTACT F

(M, η) R_η - REEB VECT ON

$R_\eta \lrcorner \eta = 1$

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GEORGES HENRI REEB 1920-1993

FRENCH MATEMATICIAN, PHD UNDER EHRESMAN

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$$X_H^C \lrcorner\, \eta = -H \qquad X_H^C \lrcorner\, d\eta = dH - R_\eta(H)\,\eta$$

$$X_H^C = Q^i \frac{\partial}{\partial q^i} + P_j \frac{\partial}{\partial p_j} + Z \frac{\partial}{\partial z}$$

$$\eta = dz - p_i dq^i \qquad\qquad d\eta = -dp_i \wedge dq^i$$

$$Z - p_i Q^i = -H \qquad -P_j dq^j + Q^i dp_i = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_j} dp^j + \frac{\partial H}{\partial z} dz - \frac{\partial H}{\partial z}\left(dz - p_i dq^i\right)$$

$$Z = p_i \frac{\partial H}{\partial p_i} - H \qquad -P_j = \frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z} \qquad Q^i = \frac{\partial H}{\partial p_i}$$

$$X_H^C = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}$$

CONTACT HAMILTONIAN VECTOR FIELDS

IN THE LITERATURE, CONTACT HAMILTONIAN DYNAMICS IS FORMULATED IN TERMS OF CONTACT FORM:

(M, η) R_η - REEB VECTOR FIELD, $H: M \longrightarrow \mathbb{R}$ HAMILTONIAN

ONE CAN CHECK THAT FOR A FIXED HAMILTONIAN $H: M \rightarrow \mathbb{R}$ AND DIFFERENT $\tilde{\eta} = f\eta$ WE GET DIFFERENT $\tilde{X}^c_H \neq X^c_H$

NO GLOBAL η $\equiv$ NO GLOBAL DYNAMICS

(FOR A GIVEN $H: M \longrightarrow \mathbb{R}$)

$$Z = p_i \frac{\partial H}{\partial p_i} - H \qquad -P_j = \frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z} \qquad Q^i = \frac{\partial H}{\partial p_i}$$

$$X^c_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \left(\frac{\partial H}{\partial q^i} + p_i \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_i} + \left(p_i \frac{\partial H}{\partial p_i} - H\right)\frac{\partial}{\partial z}$$

CONTACT HAMILTONIAN VECTOR FIELDS

- $$\mathcal{L}_{X_H^c}\eta = d\Big(\underbrace{\langle \eta, X_H^c\rangle}_{-H}\Big) + \underbrace{X_H^c \lrcorner d\eta}_{dH - R_\eta(H)\eta} = -dH + dH - R_\eta(H)\eta = -R_\eta(H)\eta$$

- $$X_H^c(H) = \langle dH, X_H^c\rangle \underset{\substack{\uparrow \\ dH = X_H^c \lrcorner d\eta + R_\eta(H)\eta}}{=} -R_\eta(H)\,H$$

CONTACT HAMILTONIAN VECTOR FIELD DOES NOT PRESERVE η NOR H

- CONTACT HAMILTONIAN VECTOR FIELD PRESERVES $\mathcal{C}$

$Y \in \mathrm{Sec}\,\mathcal{C}$ $\quad \langle \eta, Y\rangle = 0$

$$0 = X_H^c\langle \eta, Y\rangle = \underset{\substack{\| \\ 0}}{\langle \underbrace{\mathcal{L}_{X_H^c}\eta}_{\sim\eta}, Y\rangle} + \langle \eta, [X_C^H, Y]\rangle \Rightarrow [X_C^H, Y] \in \ker\eta = \mathcal{C}$$

$$(M, C) \longleftrightarrow (P, M, \tau, h, \varpi)$$

$$\mathcal{H} : P \longrightarrow \mathbb{R}$$

HOMOGENEOUS : $\mathcal{H}(h_s(p)) = s\mathcal{H}(p)$

$$\varpi(X_{\mathcal{H}}, \cdot) = d\mathcal{H}$$

HOMOGENEOUS

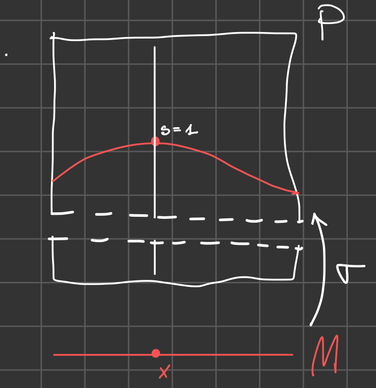
HOMOGENEOUS

INVARIANT!

$$\begin{tikzcd}
T^*P \arrow[d] & & TP \arrow[ll] \arrow[dd] \\
P \arrow[u, bend left, "d\mathcal{H}"] \arrow[urr, "X_{\mathcal{H}}"] \arrow[d] & & \\
M \arrow[rr, "X^c_{\mathcal{H}}"] & & TM
\end{tikzcd}$$

AN INVARIANT VECTOR FIELD ON P IS PROJECTABLE TO M

$$X^c_{\mathcal{H}} = T\tau(X_{\mathcal{H}})$$



$$P \simeq M \times \mathbb{R}^{\times}$$

$$\mathcal{H}(x, s) = s\mathcal{H}(\sigma(x)) = s\,H(x)$$

$$\varpi = ds \wedge \eta + s\,d\eta$$

$X_{\mathcal{H}}$ INVARIANT : $X_{\mathcal{H}} = s \cdot F(x)\frac{\partial}{\partial s} + X^c_{\mathcal{H}}$

$$d\mathcal{H} = H\,ds + s\,dH = \varpi(X_{\mathcal{H}}, \cdot) =$$

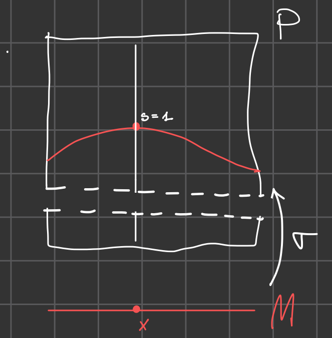
$$= sF(x)\eta - \langle \eta, X^c_{\mathcal{H}} \rangle ds + s\, X^c_{\mathcal{H}} \lrcorner\, d\eta$$

$$X^c_{\mathcal{H}} \lrcorner\, \eta = -H$$

$$dH = F(x)\eta + X^c_{\mathcal{H}} \lrcorner\, d\eta$$

CONTRACTING WITH R_η WE GET $F(x) = R_\eta(H)$

$$dH - R_\eta(H)\eta = X^c_{\mathcal{H}} \lrcorner\, d\eta$$



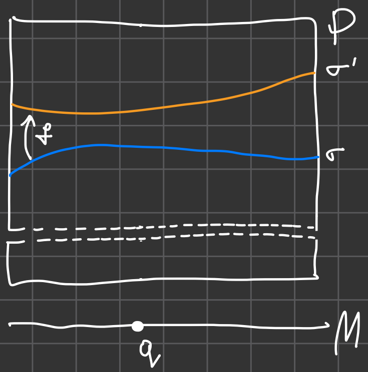
$$\mathcal{P} \simeq M \times \mathbb{R}^{\times}$$

$$\mathcal{H}(x,s) = s\mathcal{H}(\sigma(x)) = s\,H(x)$$

$$\varpi = ds \wedge \eta + s\,d\eta$$

FOR $\sigma' = f\sigma$ WE GET

$H' = fH$ $\eta' = f\eta$ AND THE SAME $X^{c}_{fH} = X_{\mathcal{H}}$



$$\sigma' = f\sigma(x)$$

$$f s' = s$$

$$\eta' = f\eta$$

$$\mathcal{H}(p) = \mathcal{H}(s'\sigma'(x)) = \mathcal{H}(s\sigma(x))$$

$$s' H'(x) = s H(x)$$

$$H'(x) = f s' H(x)$$

$$H' = fH$$

IF WE CHANGE η TO $f\eta$ AND WANT TO KEEP $X = X^{c}_{H}$
WE HAVE TO REPLACE H WITH fH

CONTACT HAMILTONIAN VECTOR FIELD CORRESPONDS TO
A HOMOGENEOUS FUNCTION ON $\mathcal{P}$

CONTACT HAMILTONIAN VECTOR FIELD CORRESPONDS TO A HOMOGENEOUS FUNCTION ON $\mathcal{P}$

IF WE NEED A HAMILTONIAN-LIKE OBJECT ON M NOT ON P IT HAS TO BE A SECTION OF CERTAIN LINE BUNDLE NOT A FUNCTION.

LINE BUNDLE

$$\begin{tikzcd} L \arrow[d, "\tau"] \\ M \end{tikzcd}$$

DUAL LINE BUNDLE

$$\begin{tikzcd} L^* \arrow[d, "\tau^*"] \\ M \arrow[u, bend left, "\sigma"] \end{tikzcd}$$

$\mathbb{R}^\times$ PRINCIPAL BUNDLE

$$\begin{tikzcd} P = L^\times \arrow[r, "H_\sigma"] \arrow[d] & \mathbb{R} \\ M \end{tikzcd}$$

SECTIONS OF L^* CORRESPOND TO HOMOGENEOUS FUNCTIONS ON $P = L^\times$

$\mathbb{R}^\times$ PRINCIPAL BUNDLE

$$\begin{tikzcd} P \arrow[d] \\ M \end{tikzcd}$$

$$\mathcal{P} = L_P^\times$$

LINE BUNDLE

$$L_P = \mathcal{P} \times \mathbb{R} / \mathbb{R}^\times$$

$$(p, r) \sim \left(sp, \frac{r}{s}\right)$$

DUAL LINE BUNDLE

$$L_P^* = \mathcal{P} \times \mathbb{R}^* / \mathbb{R}^\times$$

$$(p, r) \sim (sp, sr)$$

HOMOGENEOUS FUNCTIONS ON $\mathcal{P}$ CORRESPOND TO SECTIONS OF L_P^*

APPLICATION

$$\begin{tikzcd}
T^*P & & TP \arrow[ll] \arrow[dd] \\
P \arrow[u, bend left, "d\mathcal{H}"] \arrow[urr, "X_{\mathcal{H}}"] \arrow[d] & & \\
M \arrow[rr, "X^c_{\mathcal{H}}"] & & TM
\end{tikzcd}$$

IN CASE $M = J^1L^*$ $\left(M = T^*Q \times \mathbb{R} \right)$

$$\begin{tikzcd}
T^*T^*L^\times & & TT^*L^\times \arrow[ll] \arrow[dd] \\
T^*L^\times \arrow[u, bend left, "d\mathcal{H}"] \arrow[urr, "X_{\mathcal{H}}"] \arrow[d] & & \\
J^1L^* \arrow[rr, "X^c_{\mathcal{H}}"] & & TJ^1L^*
\end{tikzcd}$$

$M = T^*Q \times \mathbb{R}$

$L = Q \times \mathbb{R} \quad (q^i, \tau) \qquad L^\times = Q \times \mathbb{R}^\times \quad \tau \neq 0$

$L^* = Q \times \mathbb{R} \quad (q^i, z)$

$T^*L^\times = T^*(Q \times \mathbb{R}^\times) \simeq T^*Q \times \mathbb{R}^\times \times \mathbb{R} \qquad (q^i, \pi_j, \tau, z)$

$J^1L^* \simeq T^*Q \times \mathbb{R} \qquad (q^i, p_j, z)$

$$\omega_{L^\times} = d\pi_i \wedge dq^i + dz \wedge d\tau =$$
$$= d(\tau p_i) \wedge dq^i + dz \wedge d\tau =$$
$$= \tau\, dp_i \wedge dq^i + p_i\, d\tau \wedge dq^i + dz \wedge d\tau$$
$$= \underbrace{(dz - p_i dq^i)}_{\eta} \wedge d\tau + \tau\, \underbrace{dp_i \wedge dq^i}_{d\eta}$$

$$T^*L^\times \ni (q^i, \pi_j, \tau, z) \longmapsto \left(q^i, \frac{\pi_j}{\tau}, z\right)$$

STARTING FROM $H: T^*Q \times \mathbb{R} \ni (q^i, p_j, z) \longrightarrow H(q^i, p_j, z) \in \mathbb{R}$

WE CAN DEFINE $\mathcal{H}: T^*Q \times \mathbb{R} \times \mathbb{R} \ni (q^i, \pi_j, \tau, z) \longrightarrow \tau H\left(q^i, \frac{\pi_i}{\tau}, z\right) = \tau\left[H_0\left(q^i, \frac{\pi_i}{\tau}\right) - \lambda z\right] \in \mathbb{R}$

THIS ADDS VISCOSITY-LIKE FORCE TO THE USUAL MECHANICAL HAMILTONIAN SYSTEM

COORDINATE EXPRESSIONS

$$\mathcal{H}\left(x^i, \tau, \pi_j, z\right) = \tau H\left(x^i, \overset{\swarrow p_j}{\frac{\pi_j}{\tau}}, z\right)$$

ON P $$X_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} + \cancel{\tau \frac{\partial H}{\partial z}\frac{\partial}{\partial \tau} - \tau \frac{\partial H}{\partial q^j}\frac{\partial}{\partial \pi_j}} + \left(\frac{\pi_k}{\tau}\frac{\partial H}{\partial p_k} - H\right)\frac{\partial}{\partial z}$$

ON M $$X^c_{\mathcal{H}} = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \cancel{\left(\frac{\partial H}{\partial q^j} + p_j \frac{\partial H}{\partial z}\right)\frac{\partial}{\partial p_j}} + \left(p_k \frac{\partial H}{\partial p_k} - H\right)\frac{\partial}{\partial z}$$

THANK YOU!