CONTACT GEOMETRY WITH APPLICATIONS

Journal of Physics A: Mathematical and Theoretical

PAPER

A geometric approach to contact Hamiltonians and contact Hamilton–Jacobi theory

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Published 2 November 2022 • © 2022 IOP Publishing Ltd

Journal of Physics A: Mathematical and Theoretical, Volume 55, Number 43

Citation Katarzyna Grabowska and Janusz Grabowski 2022 *J. Phys. A: Math. Theor.* **55** 435204

DOI 10.1088/1751-8121/ac9adb

SUBMITTED

Contact geometric mechanics: the Tulczyjew triples

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Annali di Matematica Pura ed Applicata (1923 -)
https://doi.org/10.1007/s10231-023-01341-y

Reductions: precontact versus presymplectic

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Received: 9 February 2023 / Accepted: 28 April 2023
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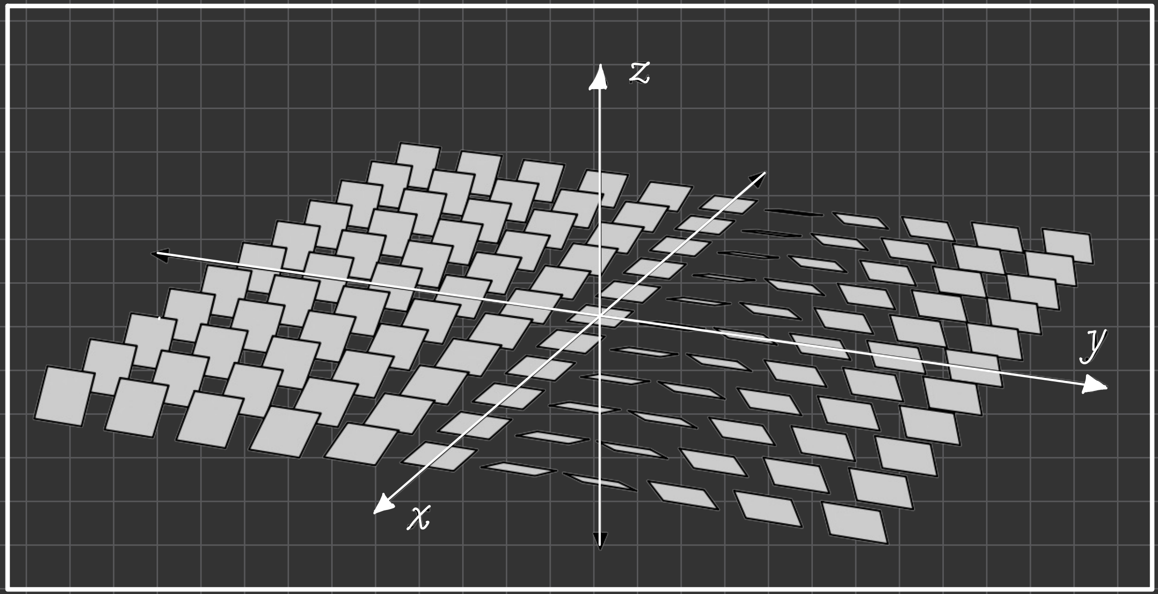
PLAN OF THE COURSE

WEDNESDAY: CONTACT GEOMETRY AND HOMOGENEOUS SYMPLECTIC GEOMETRY

THURSDAY: CONTACT GEOMETRY IN MECHANICS

FRIDAY: OTHER APPLICATIONS: HAMILTON-JACOBI THEORY, REDUCTIONS...



$$f : \mathbb{R} \ni x \longmapsto z = f(x) \in \mathbb{R}$$

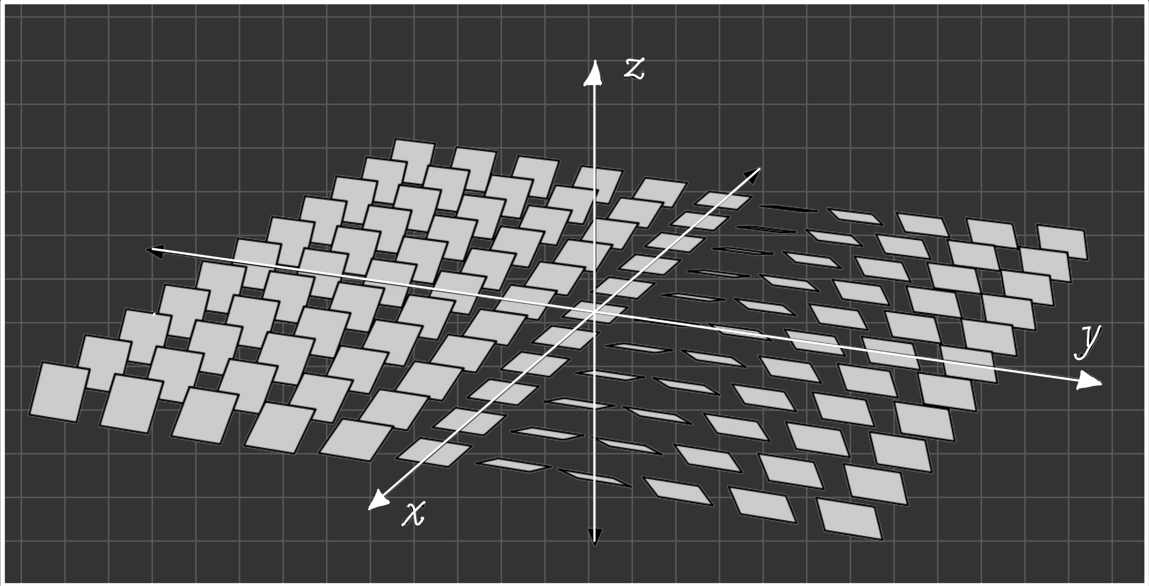
$$j^1 f(x) = \left(x,\ \overset{y}{f'(x)},\ \overset{z}{f(x)}\right) \in \mathbb{R}^3$$

$$j^1 f : \mathbb{R} \longrightarrow \mathbb{R}^3$$

$\mathcal{C}$ IS THE DISTRIBUTION OF VECTORS TANGENT TO ALL PROLONGATIONS

$$\dot{x}\frac{\partial}{\partial x} \longmapsto \dot{x}\left(\frac{\partial}{\partial x} + f''(x)\frac{\partial}{\partial y} + \underbrace{f'(x)}_{y}\frac{\partial}{\partial z}\right)$$

VARYING f WE CHANGE THIS ARBITRARY



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$$j^1 f : \mathbb{R} \longrightarrow \mathbb{R}^3$$

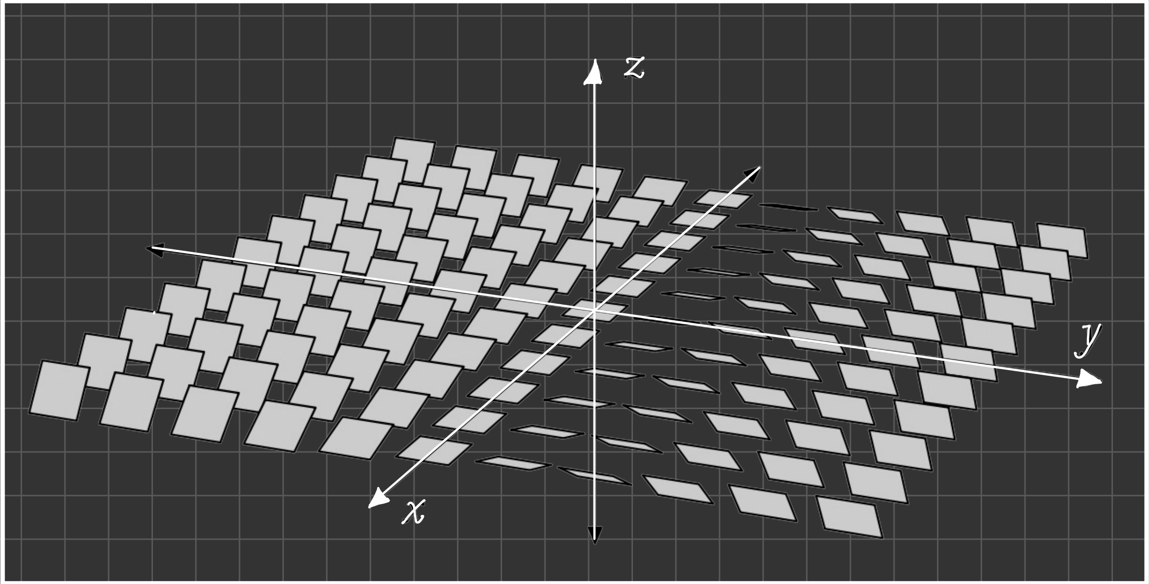
$M = \mathbb{R}^3$

$$C = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\rangle$$

$M = \mathbb{R}^3$

$$C = \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \right\rangle$$

$\mathcal{C}$ IS CALLED THE CARTAN DISTRIBUTION AND IT IS A CANONICAL STRUCTURE ON THE SPACE OF FIRST JETS



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DEFINITION: A CONTACT MANIFOLD IS A MANIFOLD M TOGETHER WITH A CORANK 1, $\underbrace{\text{MAXIMALLY NONINTEGRABLE}}$ DISTRIBUTION C

$C \subset TM \quad \dim C_x = \dim M - 1$

$L_C = TM/C$ A LINE BUNDLE $\quad \tau : TM \longrightarrow L_C$

$\nu_C : C \times_M C \longrightarrow L_C \qquad \nu_C(X, Y) = \tau([X, Y])$ (*)

$X, Y \in Sec(C)$

(*) ν_C IS DEFINED ON SECTIONS, BUT IT IS ACTUALLY A TWO FORM WITH VALUES IN L_C

$$\tau([X, fY]) = \tau\big(f[X, Y] + \underbrace{(Xf)Y}_{\text{A SECTION OF } C}\big) = f\tau([X, Y])$$

MAXIMALLY NONINTEGRABLE $\rightarrow \dim M$ ODD

$\equiv$

ν_C NONDEGENERATE

$(M, \mathcal{C})$ A CONTACT MANIFOLD

$\dim M = 2n+1$

LOCALLY: $\mathcal{C} = \ker \eta \quad \eta \in \Omega^1(M) \quad : \quad \eta \neq 0, \ \eta \wedge (d\eta)^n \neq 0$

CONTACT FORM

HOMEWORK: SHOW THAT MAXIMAL NONINTEGRABILITY IS EQUIVALENT TO THE ABOVE CONDITION

- IN OUR FIRST EXAMPLE $M = \mathbb{R}^3 \quad \mathcal{C} = \langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \rangle \quad \eta = dz - y\,dx$

- η IS NOT UNIQUE: IF $f \in C^\infty(M)$ $f \neq 0$ THEN $f\eta$ IS ANOTHER CONTACT FORM

$$(f\eta) \wedge [d(f\eta)]^n = f\eta \wedge (f\,d\eta + df \wedge \eta)^n = f\eta \wedge \left(f^n (d\eta)^n + n\, df \wedge \eta \wedge (d\eta)^{n-1}\right) = f^{n+1}\, \eta \wedge (d\eta)^n \quad \text{since } f \neq 0 \ \ldots$$

- IF M IS NOT ORIENTABLE THERE IS NO GLOBAL CONTACT FORM

  $\eta \wedge (d\eta)^n \in \Omega^{2n+1}(M)$ NONVANISHING TOP FORM

- THE CONCEPT OF A CONTACT FORM IS USEFUL (EVEN IF ONLY LOCALLY)

(M, C) A CONTACT MANIFOLD

LOCALLY: $C = \ker \eta$ $\quad \eta \in \Omega^1(M)$: $\eta \neq 0, \ \eta \wedge (d\eta)^k \neq 0$

MAXIMAL NONINTEGRABILITY CONDITION

CONTACT FORM

- η IS NOT UNIQUE: IF $f \in C^\infty(M)$ $f \neq 0$ THEN $f\eta$ IS ANOTHER CONTACT FORM
- IF M IS NOT ORIENTABLE THERE IS NO GLOBAL CONTACT FORM
- THE CONCEPT OF A CONTACT FORM IS USEFUL

THEOREM (CONTACT, DARBOUX)

IF η IS A CONTACT FORM ON $\mathcal{U} \subset M$ THEN FOR EACH $x \in \mathcal{U}$ THERE EXISTS $\mathcal{O} \subset \mathcal{U}$ AND COORDINATES (q^i, p_i, z) IN $\mathcal{O}$ SUCH THAT $\eta = dz - p_i dq^i$, $\quad C = \langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \frac{\partial}{\partial p_i} \rangle$

- ONCE WE HAVE η WE CAN DEFINE **REEB VECTOR FIELD**

$\mathcal{R} \in \mathfrak{X}(M)$: $\quad \mathcal{R} \lrcorner \eta = 1, \quad \mathcal{R} \lrcorner d\eta = 0$

STRONGLY DEPENDS ON η !!!

$\mathcal{R} = \frac{\partial}{\partial z}$ IN DARBOUX COORDINATES

$\eta \longrightarrow R_\eta$

$(f\eta) \longrightarrow R_{f\eta}$

NOT EVEN PROPORTIONAL!

$(M, \mathcal{C})$ A CONTACT MANIFOLD

MAXIMAL NONINTEGRABILITY CONDITION

LOCALLY: $\mathcal{C} = \ker \eta \qquad \eta \in \Omega^1(M) \quad : \quad \eta \neq 0, \; \eta \wedge (d\eta)^k \neq 0$

CONTACT FORM

- η IS NOT UNIQUE: IF $f \in C^\infty(M)$ $f \neq 0$ THEN $f\eta$ IS ANOTHER CONTACT FORM
- IF M IS NOT ORIENTABLE THERE IS NO GLOBAL CONTACT FORM
- THE CONCEPT OF A CONTACT FORM IS USEFUL
- DARBOUX THEOREM

- ONCE WE HAVE η WE CAN DEFINE **REEB VECTOR FIELD**

STRONGLY DEPENDS ON η !!!

$$R \in \mathfrak{X}(M) : \qquad R_\eta \lrcorner \, \eta = 1, \qquad R_\eta \lrcorner \, d\eta = 0$$

$R_\eta = \frac{\partial}{\partial z}$ IN DARBOUX COORDINATES

$\eta \longrightarrow R_\eta$

$(f\eta) \longrightarrow R_{f\eta}$

NOT EVEN PROPORTIONAL!

HOMEWORK: SHOW THAT FOR A GIVEN $X_q \in T_qM$ $X_q \notin \mathcal{C}_q$ THERE EXISTS A LOCAL η SUCH THAT

$$X_q = R_\eta(q)$$

EXAMPLE – JETS OF FUNCTIONS

WE THINK OF FUNCTIONS ON A MANIFOLD Q AS SECTIONS OF THE TRIVIAL BUNDLE
$$\begin{tikzcd} Q\times\mathbb{R} \arrow[d, shift left] \\ Q \arrow[u, shift left, "f"] \end{tikzcd}$$

$$\begin{tikzcd} Q\times\mathbb{R} \arrow[d, shift left] \\ Q \arrow[u, shift left, "f"] \end{tikzcd}$$

(q^i, p_i) $\quad z$

$$M = J^1(Q\times\mathbb{R}) \simeq T^*Q\times\mathbb{R}$$

$$\eta = dz - \theta_Q = dz - p_i\, dq^i$$

$$TM \supset C = \left\langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z}, \frac{\partial}{\partial p_i} \right\rangle$$

C IS SPANNED BY VECTORS TANGENT TO IMAGES OF $j^1 f : Q \longrightarrow J^1(Q\times\mathbb{R})$

$p_i \quad z$

$$j^1 f : (q^i) \longmapsto \left(q^i, \frac{\partial f}{\partial q_i}, f(q)\right)$$

$$Tj^1 f\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial z}$$

EXAMPLE – JETS OF FUNCTIONS

WE THINK OF FUNCTIONS ON A MANIFOLD Q AS SECTIONS OF THE TRIVIAL BUNDLE $\begin{array}{c} Q\times\mathbb{R} \\ \downarrow \uparrow f \\ Q \end{array}$

$$\begin{array}{c} Q\times\mathbb{R} \\ \downarrow \uparrow f \\ Q \end{array} \qquad M = J^1(Q\times\mathbb{R}) \simeq \underbrace{T^*Q}_{(q^i,p_i)} \times \underbrace{\mathbb{R}}_{z} \qquad \eta = dz - \theta_Q = dz - p_i\,dq^i$$

$$TM \supset C = \left\langle \frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial z},\ \frac{\partial}{\partial p_i} \right\rangle$$

C IS SPANNED BY VECTORS TANGENT TO IMAGES OF $j^1f : Q \longrightarrow J^1(Q\times\mathbb{R})$

$$j^1f : (q^i) \longmapsto \Big(q^i,\ \underbrace{\frac{\partial f}{\partial q_i}}_{p_i},\ \underbrace{f(q)}_{z}\Big)$$

$$j^1f\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial z}$$

EXAMPLE – JETS OF SECTIONS

INSTEAD OF FUNCTIONS ON Q WE TAKE SECTIONS OF A (POSSIBLY NONTRIVIAL) LINE BUNDLE. THE SIMPLEST NONTRIVIAL EXAMPLE IS THE MÖBIUS BAND

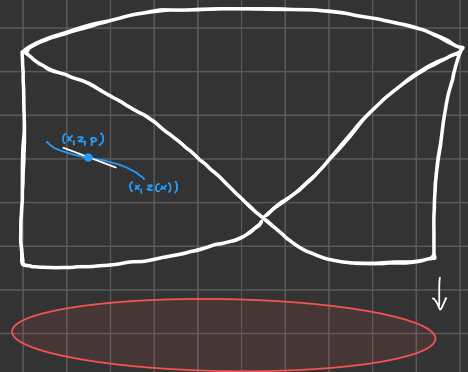
$$\mathbb{Z}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(k,x,y) \longmapsto (x+k, (-1)^k y)$$

$$B = \mathbb{R}^2/\mathbb{Z} \qquad S^1 = \mathbb{R}/\mathbb{Z}$$

$B \longrightarrow S^1$ LINE BUNDLE

$B^* \longrightarrow S^1$ DUAL LINE BUNDLE



EXAMPLE – JETS OF FUNCTIONS

WE THINK OF FUNCTIONS ON A MANIFOLD Q AS SECTIONS OF THE TRIVIAL BUNDLE $\begin{array}{c} Q\times\mathbb{R} \\ \downarrow \uparrow f \\ Q \end{array}$

$$\begin{array}{c} Q\times\mathbb{R} \\ \downarrow \;\uparrow f \\ Q \end{array}$$

$$M = J^1(\underbrace{Q\times\mathbb{R}}) \simeq \overbrace{T^*Q}^{(q^i,p_i)} \times \overbrace{\mathbb{R}}^{z}$$

$$\eta = dz - \theta_Q = dz - p_i\, dq^i$$

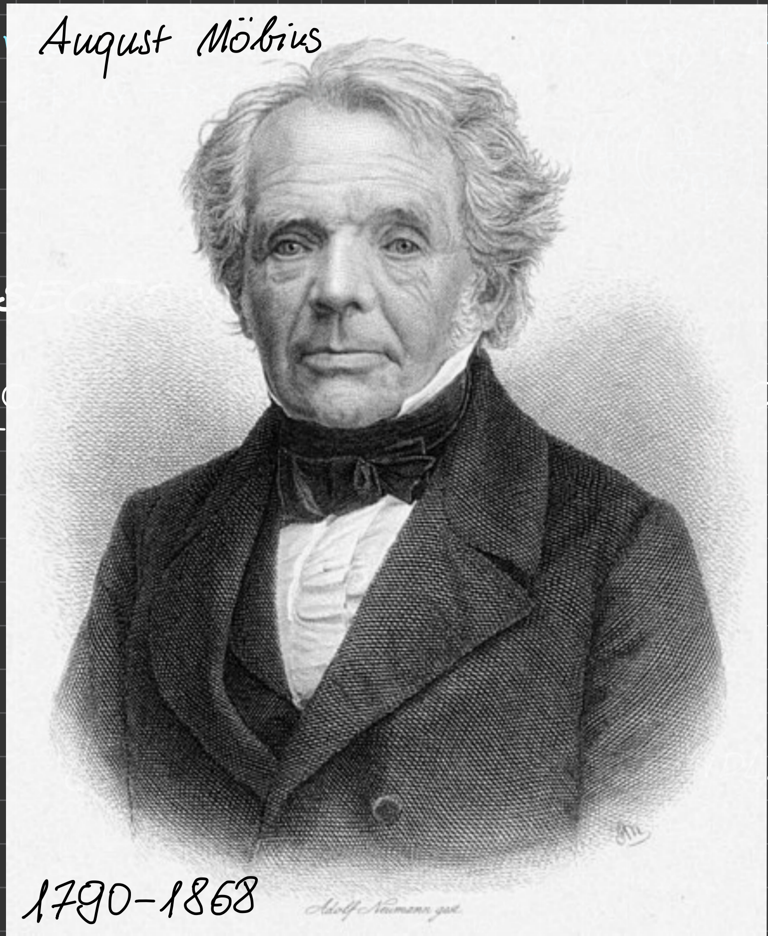
$$TM \supset C = \left\langle \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial z},\ \frac{\partial}{\partial p_i} \right\rangle$$

C IS SPANNED BY ...

TO IMAGES OF ...

$$\longrightarrow \Big(q^i, \underbrace{\frac{\partial f}{\partial q_i}}_{p_i}, \underbrace{f(q)}_{z}\Big)$$

$$= \frac{\partial}{\partial q^i} + \frac{\partial f}{\partial q_i}\frac{\partial}{\partial z}$$



EXAMPLE – JETS OF S...

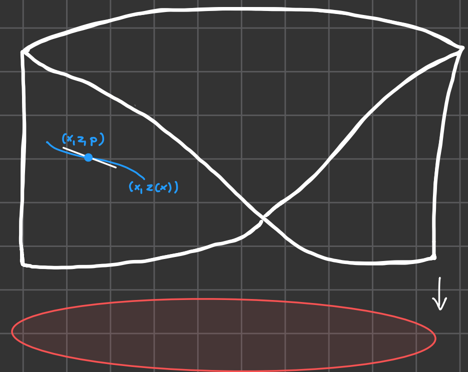
INSTEAD OF FUNCTIONS ... OF A (POSSIBLY NONTRIVIAL) LINE BUNDLE. THE SIMPLEST ... MÖBIUS BAND

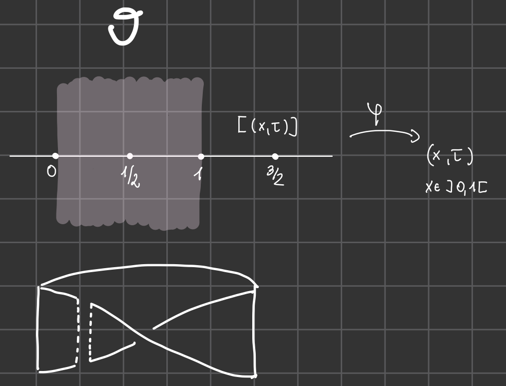
$$\mathbb{Z}\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

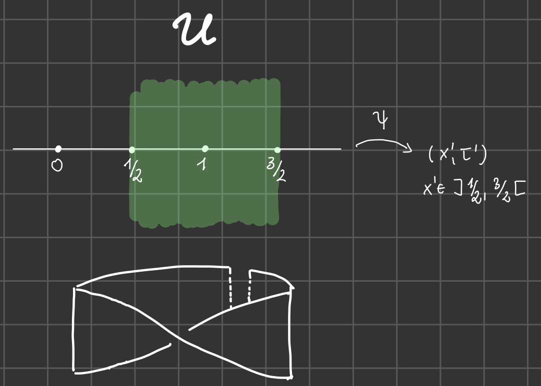
$$(k,x,y) \longmapsto (x+k, (-1)^k y)$$

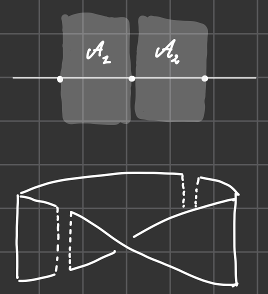
$$B = \mathbb{R}^2/\mathbb{Z} \qquad S^1 = \mathbb{R}/\mathbb{Z}$$

E









$$\psi \circ \varphi^{-1}(x,\tau) = \begin{cases} (x,\tau) & \frac{1}{2} < x < 1 \\ (x+1, -\tau) & 0 < x < \frac{1}{2} \end{cases}$$

J^1B^* IS GIVEN BY TWO CHARTS $\bar{\mathcal{O}}, \bar{\mathcal{U}}$ OVER $\mathcal{O}, \mathcal{U}$ RESPECTIVELY

$\bar{\varphi}$ (x, z, p) FOR $\bar{\mathcal{O}}$

$\bar{\psi}$ (x', z', p') FOR $\bar{\mathcal{U}}$

WITH TRANSFORMATIONS

$$\bar{\psi} \circ \bar{\varphi}^{-1}(x,z,p) = \begin{cases} (x, z, p) & \frac{1}{2} < x < 1 \\ (x+1, -z, -p) & 0 < x < \frac{1}{2} \end{cases}$$

CONTACT DISTRIBUTION

$$C = \left\langle \frac{\partial}{\partial p}, \frac{\partial}{\partial x} + p\frac{\partial}{\partial z} \right\rangle = \left\langle \frac{\partial}{\partial p'}, \frac{\partial}{\partial x'} + p'\frac{\partial}{\partial z'} \right\rangle$$

OVER A_1 $p = -p'$

$$\frac{\partial}{\partial p} = -\frac{\partial}{\partial p'} \qquad \frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad \frac{\partial}{\partial z} = -\frac{\partial}{\partial z'}$$

$$\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} = \frac{\partial}{\partial x'} + p'\frac{\partial}{\partial z'}$$

OVER A_2

...

$$\frac{\partial}{\partial x} + p\frac{\partial}{\partial z} = \frac{\partial}{\partial x'} + p'\frac{\partial}{\partial z'}$$

DISTRIBUTION C IS WELL DEFINED

CONTACT FORM

$$C = \ker \eta = \ker \eta'$$

$$\eta = dz - p\,dx \qquad \eta' = dz' - p'dx'$$

OVER A_1 $\eta = -\eta'$

OVER A_2 $\eta = \eta'$

NO GLOBAL CONTACT FORM!

$$(M, C) \longleftrightarrow (M, \eta)$$

WHY IS IT A PROBLEM?

IN THE LITERATURE CONTACT GEOMETRY IS MOSTLY DONE IN THE LANGUAGE OF CONTACT FORMS! MANY CONSTRUCTIONS STRONGLY DEPEND ON A CHOSEN FORM

WE NEED A NEW LANGUAGE SUITABLE FOR GENERAL CONTACT MANIFOLDS!

SYMPLECTIC PRINCIPAL BUNDLES

$\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ MULTIPLICATIVE GROUP OF REALS $\mathbb{R}^\times = GL(1, \mathbb{R})$

DEFINITION: A SYMPLECTIC $\mathbb{R}^\times$-PRINCIPAL BUNDLE IS AN $\mathbb{R}^\times$-PRINCIPAL BUNDLE $P \longrightarrow M$ WITH A SYMPLECTIC FORM ω ON P SUCH THAT $h_s^* \omega = s\omega$

SMOOTH FIBRE BUNDLE

$$P \circlearrowleft \mathbb{R}^\times, \quad P \xrightarrow{\tau} M$$

$$(\overbrace{P, M, \tau}, h, \omega)$$

THE HOMOGENEOUS SYMPLECTIC FORM

$$h_s^* \omega = s\omega$$

$$h : \mathbb{R}^\times \times P \longrightarrow P$$

PRINCIPAL ACTION

$$h_s(p) = h(s, p)$$

EXAMPLE

$P = (T^*Q)^\times \qquad \tau : (T^*Q)^\times \longrightarrow \mathbb{P}T^*Q$

$h(s, \alpha) = s\alpha \qquad \omega_Q = dp_i \wedge dq^i$

$$h_s^* \omega_Q = d(sp_i) \wedge dq^i = s\,dp_i \wedge dq^i = s\omega_Q$$

EVERY CONTACT (M, C) DEFINES (P, M, τ, h, ω)

$C \subset TM \quad \dim M = 2n+1$

$\dim C_q = 2n$

$C^0 \subset T^*M \quad \dim (C^0_q) = 1$

$\dim (C^0) = 2n+1+1 = 2n+2$

$C^0 = \langle \eta \rangle$ ANY CONTACT FORM

- $P = (C^0)^\times \qquad P \subset T^*M$
- $\tau = \pi_M\big|_P$
- h MULTIPLICATION OF COVECTORS BY NUMBERS
- $\omega = \omega_M\big|_P$

EVERY CONTACT (M, C) DEFINES (P, M, τ, h, ω)

$C \subset TM \quad \dim M = 2n+1$

$\dim C_q = 2n$

$C^\circ \subset T^*M \quad \dim (C^\circ_q) = 1$

$\dim (C^\circ) = 2n+1+1 = 2n+2$

$C^\circ = \langle \eta \rangle$ ANY CONTACT FORM

- $P = (C^\circ)^\times$
- $\tau = \pi_M|_P$
- h MULTIPLICATION OF COVECTORS BY NUMBERS
- $\omega = \omega_M|_P$

ω_M IS A LINEAR TWO FORM THEREFORE $\omega_M|_P$ IS HOMOGENEOUS

P IS A SYMPLECTIC SUBMANIFOLD:

$C^\circ = \langle \eta \rangle$

$\mathcal{O} \subset M$

$$\mathbb{R}^\times \times \mathcal{O} \xrightarrow{I_\eta} P_\mathcal{O} \subset T^*\mathcal{O}$$

$$(s, q) \longmapsto s\eta(q)$$

$$I_\eta^* \omega_M = I_\eta^* d\theta_M = d\left(I_\eta^* \theta_M\right) = d\left(s\eta^* \theta_M\right) = d\left(s\eta\right) = ds \wedge \eta + s\,d\eta =: \omega_\eta$$

$$(\omega_\eta)^{n+1} = (ds\wedge\eta + s\,d\eta)^{n+1} = \underbrace{s^{n+1}\,d\eta^{n+1}}_{=0} + s\,ds\wedge\eta\wedge d\eta^n = s\,ds\wedge\underbrace{\eta\wedge d\eta^n}_{\neq 0} \neq 0$$

CLOSED

NONDEGENERATE

SYMPLECTIC

because $\eta\wedge d\eta^n$ is a top form on $\mathcal{O}\subset M$
therefore $(d\eta)^{n+1} = d(\eta\wedge d\eta^n) = 0$

EVERY CONTACT (M, C) DEFINES (P, M, τ, h, ω)

$C \subset TM \quad \dim M = 2n+1$

$\dim C_q = 2n$

$C^\circ \subset T^*M \quad \dim (C^\circ_q) = 1$

$\dim (C^\circ) = 2n+1+1 = 2n+2$

$C^\circ = \langle \eta \rangle$ ← ANY CONTACT FORM

- $P = (C^\circ)^\times$
- $\tau = \pi_M \big|_P$
- h MULTIPLICATION OF COVECTORS BY NUMBERS
- $\omega = \omega_M \big|_P$

IF η IS GLOBAL: WE CAN CONSTRUCT ANOTHER SYMPLECTIC COVER

$P_0 = M \times \mathbb{R} \quad \omega = d(e^t \eta) = e^t d\eta + e^t dt \wedge d\eta$

(q, t)

P IS TRIVIALISABLE $\quad P \simeq M \times \mathbb{R}^\times \quad P_0 \simeq M \times \mathbb{R}^\times_{>0} \subset P$

THANK YOU!