Introduction to Infinite-Dimensional Lie Groups III

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29.06.2023

Overview

- Examples:
  - Lie groups $\mathrm{Diff}(M)$ of smooth diffeomorphisms
  - Direct limit groups $G = \bigcup_{n\in\mathbb{N}} G_n$ with $G_1 \subseteq G_2 \subseteq \cdots$, e.g.

$$\mathrm{GL}_\infty(\mathbb{R}) = \bigcup_{n\in\mathbb{N}} \mathrm{GL}_n(\mathbb{R})$$

identify $A \in \mathrm{GL}_n(\mathbb{R})$ with $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$.

- Lie algebra $\mathfrak{g} := L(G)$ and $\exp_G \colon \mathfrak{g} \to G$
- Locally exponential Lie groups and BCH-Lie groups
- From Lie algebra to Lie group: Regularity

§0 Repetition

If G is a Lie group modelled on a locally convex space E and $\theta\colon V \to U$ a local parametrisation with $e \in U$,

[a C^∞-diffeomorphism from an open set $V \subseteq E$ onto an open set $U \subseteq G$]

then $V \to gU$, $x \mapsto g\theta(x)$ is a local parametrization for each $g \in G$, with $g \in gU$.

Hence:

Main point for Lie group structure is to know a local parametrization around e (or the corresponding chart $\theta^{-1}\colon U \to V$).

For example, if G is a Lie group, M a compact smooth manifold and $n \in \mathbb{N}_0 \cup \{\infty\}$, then

$$\theta_*\colon C^n(M,V) \to C^n(M,U) \subseteq C^n(M,G), \quad \gamma \mapsto \theta \circ \gamma$$

is a local parametrization around $x \mapsto e$ for a local parametrization $\theta\colon V \to U$ of G with $e \in U$.

§1 The diffeomorphism group of $[0,1]$

Let $\mathrm{Diff}_+([0,1])$ be the group of all C^∞-diffeomorphisms $\gamma\colon [0,1] \to [0,1]$ which are orientation-preserving. Thus $\gamma \in C^\infty([0,1],\mathbb{R})$,

$$\gamma(0) = 0, \quad \gamma(1) = 1 \quad \text{and } \gamma'(t) > 0 \text{ for all } t \in [0,1].$$

Now

$$C^\infty_\partial([0,1],\mathbb{R}) := \{\gamma \in C^\infty([0,1],\mathbb{R})\colon \gamma(0) = \gamma(1) = 0\}$$

is a closed vector subspace of $C^\infty([0,1],\mathbb{R})$. The set

$$\Omega := \{\gamma \in C^\infty_\partial([0,1],\mathbb{R})\colon \gamma'(t) > -1 \text{ for all } t \in [0,1]\}$$

is open in $C^\infty_\partial([0,1],\mathbb{R})$ and convex (hence contractible). The map

$$\phi\colon \mathrm{Diff}_+([0,1]) \to \Omega, \quad \gamma \mapsto \gamma - \mathrm{id}_{[0,1]}$$

is a bijection.

We give $\mathrm{Diff}_+([0,1])$ the smooth manifold structure making ϕ a C^∞-diffeomorphism. Then ϕ is a global chart for $\mathrm{Diff}_+([0,1])$ and

$$\theta := \phi^{-1}\colon \Omega \to \mathrm{Diff}_+([0,1]), \quad \gamma \mapsto \mathrm{id}_{[0,1]} + \gamma$$

a global parametrization of $\mathrm{Diff}_+([0,1])$. Give Ω the group multiplication $*$ making ϕ an isomorphism. Thus

$$\begin{aligned}\gamma * \eta &= (\mathrm{id} + \gamma) \circ (\mathrm{id} + \eta) - \mathrm{id} \\ &= \eta + \gamma \circ (\mathrm{id}_{[0,1]} + \eta)\,.\end{aligned}$$

This is explicit enough to calculate directional derivatives by hand and see that $*$ is smooth. Also the inversion map is smooth and thus Ω (and $\mathrm{Diff}_+([0,1])$) are Lie groups (see G.-Neeb 2017, also G. 2023)

§2 Diffeomorphism groups of compact manifolds

If M is a compact smooth manifold, consider the group $\mathrm{Diff}(M)$ of all C^∞-diffeomorphisms $\gamma\colon M \to M$. The vector space $\mathcal{V}(M)$ of all smooth vector fields $X\colon M \to TM$ on M can be made a Fréchet space, similar to the compact-open C^∞-topology on function spaces from the last lecture.

Fact (cf. Michor '80, Hamilton '20, Milnor '84, Kriegl-Michor '97)

$\mathrm{Diff}(M)$ can be made a Lie group modelled on $\mathcal{V}(M)$. The Lie group structure is uniquely determined by the following exponential law: For each smooth manifold N modelled on a locally convex space, a map

$$f\colon N \to \mathrm{Diff}(M)$$

is smooth if and only if the map

$$f^\wedge\colon N \times M \to M, \quad (x,y) \mapsto f(x)(y)$$

is smooth.

The main point is to describe a local parametrization around the neutral element id_M. Pick a Riemannian metric g on M and let

$$\exp_g \colon TM \to M$$

be the Riemannian exponential map (taking $v \in T_pM$ to the value at time $t = 1$ of the geodesic starting with velocity v at the position p at time $t = 0$). Consider the map

$$(\exp_g)_* \colon \mathcal{V}(M) \to C^\infty(M, M), \quad X \mapsto \exp_g \circ X \, .$$

One can show that

$$U := (\exp_g)_*(V) \subseteq \mathrm{Diff}(M)$$

for some open 0-neighbourhood $V \subseteq \mathcal{V}(M)$ and that

$$(\exp_g)_*|_V \colon V \to U \subseteq \mathrm{Diff}(M)$$

can be used as a local parametrization of $\mathrm{Diff}(M)$ around id_M, for V sufficiently small.

§3 Direct limits of finite-dimensional Lie groups

Consider finite-dimensional Lie groups $G_1 \subseteq G_2 \subseteq \cdots$

such that the inclusion map $G_n \to G_{n+1}$ is a continuous (and hence smooth) group homomorphism. Then $G := \bigcup_{n\in\mathbb{N}} G_n$ is a group: Given $x, y \in G$, there exist $n, m \in \mathbb{N}$ with $x \in G_n$ and $y \in G_m$. Multiply x and y in $G_{\max\{n,m\}}$.

Consider a real vector space
$$\mathbb{R}^\infty := \bigoplus_{n\in\mathbb{N}} \mathbb{R}$$
of countable, infinite dimension. Make $\mathbb{R}^\infty$ a locally convex topological vector space using the set of **all** seminorms on $\mathbb{R}^\infty$.

Fact (G. 2005)

There is a unique Lie group structure on $G = \bigcup_{n\in\mathbb{N}} G_n$ modelled on $\mathbb{R}^m$ for some $m \in \mathbb{N}_0$ or on $\mathbb{R}^\infty$ such that, for each smooth manifold M modelled on a locally convex space and map $f: G \to M$, we have: f is smooth $\Leftrightarrow$ $f|_{G_n}$ is smooth for all $n \in \mathbb{N}$.

Thus $G = \varinjlim G_n$ as a C^∞-manifold; hence also as a Lie group.

To get examples, identify $A \in \mathrm{GL}_n(\mathbb{R})$ with

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R}). \qquad \text{Then}$$

$\mathrm{GL}_1(\mathbb{R}) \subseteq \mathrm{GL}_2(\mathbb{R}) \subseteq \cdots,$
$\mathrm{SL}_1(\mathbb{R}) \subseteq \mathrm{SL}_1(\mathbb{R}) \subseteq \cdots,$
$O_1(\mathbb{R}) \subseteq O_2(\mathbb{R}) \subseteq \cdots,$
$SO_1(\mathbb{R}) \subseteq SO_2(\mathbb{R}) \subseteq \cdots,$
$\mathrm{GL}_1(\mathbb{C}) \subseteq \mathrm{GL}_2(\mathbb{C}) \subseteq \cdots,$
$U_1(\mathbb{C}) \subseteq U_2(\mathbb{C}) \subseteq \cdots,$
$SU_1(\mathbb{C}) \subseteq SU_2(\mathbb{C}) \subseteq \cdots$

We can form the direct limit Lie groups

$$\mathrm{GL}_\infty(\mathbb{R}) := \bigcup_{n\in\mathbb{N}} \mathrm{GL}_n(\mathbb{R}),\ \mathrm{SL}_\infty(\mathbb{R}) := \bigcup_{n\in\mathbb{N}} \mathrm{SL}_n(\mathbb{R}),\ O_\infty(\mathbb{R}) := \bigcup_{n\in\mathbb{N}} O_n(\mathbb{R}),$$

$$SO_\infty(\mathbb{R}) := \bigcup_{n\in\mathbb{N}} SO_n(\mathbb{R}),\ \mathrm{GL}_\infty(\mathbb{C}) := \bigcup_{n\in\mathbb{N}} \mathrm{GL}_n(\mathbb{C}),\ U_\infty(\mathbb{C}) := \bigcup_{n\in\mathbb{N}} U_n(\mathbb{C}),$$

and $SU_\infty(\mathbb{C}) := \bigcup_{n\in\mathbb{N}} SU_n(\mathbb{C})$.

[For $\mathbb{R}^\infty$, cf. Bisgaard '93, Kakutani-Klee '63, Hirai et al. 2001).]

§4 Lie algebra and exponential map

If M is a smooth manifold modelled on a locally convex space E, let **tangent vectors** at $p \in M$ be equivalence classes $[\gamma]$ of smooth curves

$$\gamma\colon\ ]-\varepsilon,\varepsilon[\ \to M \qquad \text{with } \gamma(0) = p,$$

where $\gamma \sim \eta$ if $(\phi \circ \gamma)'(0) = (\phi \circ \eta)'(0)$ for some (and hence each) chart $\phi\colon U \to V \subseteq E$ of M with $p \in U$.

Let T_pM be the set of all tangent vectors at $p \in M$ and $TM := \bigcup_{p \in M} T_pM$.

The map

$$h_\phi\colon T_pM \to E, \quad [\gamma] \mapsto (\phi \circ \gamma)'(0)$$

is a bijection with inverse $v \mapsto [t \mapsto \phi^{-1}(\phi(p) + tv)]$. Give T_pM the topological vector space structure making h_ϕ an isomorphism of topological vector spaces. The map

$$\pi_{TM}\colon TM \to M, \quad T_pM \ni v \mapsto p$$

(i.e., $[\gamma] \mapsto \gamma(0)$) is well defined.

For each smooth map $f\colon M \to N$ between manifolds modelled on locally convex spaces, the map

$$T_p f\colon T_p M \to T_{f(p)} N, \quad [\gamma] \mapsto [f \circ \gamma]$$

is continuous and linear. Define

$$Tf\colon TM \to TN, \quad T_p M \ni v \mapsto T_p f(v)\,.$$

Taking $\phi = \mathrm{id}_V$, see that for an open subset $V \subseteq E$, the map

$$TV \to V \times E, \quad [\gamma] \mapsto (\gamma(0), \gamma'(0))$$

is a bijection. Identify TV with $V \times E$. Give TM the smooth manifold structure modelled on $E \times E$ turning TU into an open subset and making

$$T\phi\colon TU \to TV = V \times E$$

a chart for TM, for each chart $\phi\colon U \to V \subseteq E$ of M.

A **smooth vector field** on M is a smooth map $X\colon M \to TM$ such that $X(p) \in T_pM$ for all $p \in M$, i.e.,

$$\pi_{TM} \circ X = \mathrm{id}_M \; .$$

Given a smooth function $f\colon M \to \mathbb{R}$, get a smooth function $\mathcal{L}_X(f) := X.f := df \circ X\colon M \to \mathbb{R}$, where $df\colon TM \to \mathbb{R}$ is the second component of

$$Tf\colon TM \to T(\mathbb{R}) = \mathbb{R} \times \mathbb{R} \, .$$

Vector fields can be added and multiplied with real numbers pointwise. On the vector space $\mathcal{V}(M)$ of all smooth vector fields, there is a unique Lie bracket $[.,.]$ such that

$$\mathcal{L}_{[X,Y]|_U} = \mathcal{L}_{X|_U} \circ \mathcal{L}_{Y|_U} - \mathcal{L}_{Y|_U} \circ \mathcal{L}_{X|_U}$$

for all $X, Y \in \mathcal{V}(M)$ and open subsets $U \subseteq M$.

For $g \in G$, consider the left translation $L_g\colon G \to G$, $x \mapsto gx$. Then

$$G \times TG \to TG, \quad (g, v) \mapsto g.v := TL_g(v)$$

is a smooth left action of G on TG. Let $\mathcal{V}_\ell \subseteq \mathcal{V}(G)$ be the set of all vector fields $X\colon G \to TG$ which are **left invariant** in the sense that

$$X(g) = g.X(e) \quad \text{for all } g \in G.$$

As in finite dimensions, one finds that $\mathcal{V}_\ell$ is a Lie subalgebra of $\mathcal{V}(G)$. The map

$$\mathcal{V}_\ell \to T_e(G), \quad X \mapsto X(e)$$

is an isomorphism of vector spaces. We give T_eG the Lie bracket turning the latter map into an isomorphism of Lie algebras. One can show that the Lie bracket is continuous; we write

$$L(G) := T_eG$$

for T_eG endowed with the topological Lie algebra structure just described.

As in the finite-dimensional case, one finds that the continuous linear map

$$L(f) := T_e f\colon T_e G \to T_e H$$

is a Lie algebra homomorphism for each smooth homomorphism $f\colon G \to H$ between Lie groups.

If $\gamma\colon \mathbb{R} \to M$ is smooth, define $\dot{\gamma}(t) := T\gamma(t,1) = [s \mapsto \gamma(t+s)]$ for $t \in \mathbb{R}$. We shall see later:

Let G be a Lie group modelled on a locally convex space. For $v \in \mathfrak{g} := L(G)$, there is at most one smooth group homomorphism

$$\gamma_v\colon (\mathbb{R},+) \to G$$

such that $\dot{\gamma}_v(0) = v$. If γ_v exists for all $v \in \mathfrak{g}$, we say that G **has an exponential function** and define

$$\exp_G\colon \mathfrak{g} \to G, \quad v \mapsto \gamma_v(1)\,.$$

Then $\exp_G(tv) = \gamma_v(t)$.

If $f\colon G \to H$ is a smooth group homomorphism and both G and H have an exponential function, then

$$\exp_H \circ L(f) = f \circ \exp_G$$

("naturality of exp")

Open Problem

Does every Lie group G modelled on a **sequentially complete** locally convex space have an exponential function?

No counterexamples known!

Remark. In the absence of sequential completeness of the modelling space, Lie groups without an exponential function do exist.

Definition

A Lie group G is called **locally exponential** if it has an exponential function and the latter is a local C^∞-diffeomorphism at 0.

Thus, there is an open 0-neighbourhood $V \subseteq \mathfrak{g} := L(G)$ such that $\exp_G(V)$ is open in G and $\exp_G|_V \colon V \to \exp_G(V)$ is a C^∞-diffeomorphism.

A real analytic Lie group G is called a real BCH-Lie group if it has an exponential function which is a local real analytic diffeomorphism at 0.

For small $x, y \in \mathfrak{g} := L(G)$,

$$x * y = \exp_G^{-1}(\exp_G(x)\exp_G(y))$$

then must be given by the Baker-Campbell-Hausdorff series,

$$x * y = x + y + \frac{1}{2}[x, y] + \frac{1}{12}([x, [x, y]] + [y, [y, x]]) + \cdots$$

Complex BCH-Lie groups are defined analogously, replacing the word "real analytic" with "complex analytic."

Example (see G. 2002a)

For each sequentially complete continuous inverse algebra A over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, the unit group $A^\times$ is a BCH-Lie group over $\mathbb{K}$.

The exponential function is given by the exponential series, $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. In the complex case, it is obtained via holomorphic functional calculus. Classical example (see, e.g., Bourbaki):

Every $\mathbb{K}$-analytic Banach-Lie group is a BCH-Lie group over $\mathbb{K}$.

For each BCH-Lie group G over $\mathbb{K}$, $n \in \mathbb{N}$ and compact C^∞-manifold M, also $C^n(M, G)$ is a BCH-Lie group (G. 2002b).

Using $\mathfrak{g} := L(G)$, the Lie algebra of $C^n(M, G)$ is $C^n(M, \mathfrak{g})$ and

$$(\exp_G)_*\colon C^n(M, \mathfrak{g}) \to C^n(M, G), \quad \gamma \mapsto \exp_G \circ \gamma$$

is the exponential function for $C^n(M, G)$. It is a local $\mathbb{K}$-analytic diffeomorphism as $\exp_G|_V$ is a local parametrization of G.

for an open 0-neighbourhood $V \subseteq \mathfrak{g}$ and hence

$$(\exp_G|_V)_*\colon C^n(M,V) \to C^n(M,\exp_G(V)) \subseteq C^n(M,G)$$

a local parametrization for $C^n(M,G)$ for small V, by contstruction of the Lie group structure on mapping groups.

Example

The diffeomorphism group $\mathrm{Diff}(\mathbb{S}_1)$ of the circle has a smooth exponential map

$$\mathcal{V}(\mathbb{S}_1) \to \mathrm{Diff}(\mathbb{S}_1), \quad X \mapsto \mathrm{Fl}^X_{t=1},$$

but it is not locally exponential.

The image of the exponential map is not an identity neighbourhood; Grabowski '88 found an injective smooth curve

$$\gamma\colon [0,1[\to \mathrm{Diff}(\mathbb{S}_1)$$

with $\gamma(0) = \mathrm{id}_{\mathbb{S}_1}$ and $\gamma(t)$ outside the exponential image for all $t \in]0,1[$ (with $\gamma(]0,1[)$ a free generating set for a free group).

Example

Direct limit Lie groups $G = \varinjlim G_n$ of finite-dimensional Lie groups need not be locally exponential.

They have a smooth exponential function but for an example in G. 2003, $\exp_G$ is neither injetive on any 0-neighbourhood nor the exponential image an identity neighbourhood.

Remark. Sufficient conditions are known which ensure that $\bigcup_{n\in\mathbb{N}} G_n$ is BCH, even when $G_1 \subseteq G_2 \subseteq \cdots$ are Banach–Lie groups which may be infinite-dimensional (Dahmen 2014).

The exponential function is of limited use if $\exp_G$ fails to be a local diffeomorphism at 0. Replacement: regularity.

§6 Regularity

A Lie group G with Lie algebra $\mathfrak{g} := L(G)$ is called **semiregular** if the initial value problem

$$\dot{\gamma}(t) = \gamma(t).\eta(t), \quad \gamma(0) = e$$

has a (necessarily unique) smooth solution $\gamma\colon [0,1] \to G$ for each smooth curve $\eta\colon [0,1] \to \mathfrak{g}$. If, moreover, the map

$$\mathrm{Evol}\colon C^\infty([0,1],\mathfrak{g}) \to C([0,1],G), \quad \eta \mapsto \gamma$$

is smooth, then G is called *regular*.

Use left action $G \times TG \to TG$ here, as above.

Equivalently, require Evol is smooth to $C^\infty([0,1],G)$, or require the time-1-map evol: $C^\infty([0,1],\mathfrak{g}) \to G$, $\eta \mapsto \mathrm{Evol}(\eta)(1)$ is smooth.

For Lie groups modelled on sequentially complete spaces, regularity was introduced by Milnor, simplifying a concept by Omori and coauthors.

For example, $\gamma_v = \mathrm{Evol}(v)$ and $\exp_G(v) = \mathrm{evol}(v)$ for $v \in \mathfrak{g}$ and the constant curve $t \mapsto v$.

The idea of regularity is to work not only with the exponential function – which corresponds to evolutions of constant curves – but to work with $\mathrm{Evol}(\eta)$ for arbitrary, not necessarily constant functions $\eta \in C^\infty([0,1], \mathfrak{g})$.

Then much works as in finite-dimensional Lie theory. For instance:

Theorem (Milnor '84)

Let G and H be Lie groups and $\beta\colon L(G) \to L(H)$ be a continuous Lie algebra homomorphism. If G is simply connected and H is regular, then there exists a unique smooth group homomorphism $\alpha\colon G \to H$ such that $L(\alpha) = \beta$.

Remark. An analogous conclusion (with α analytic) holds without the assumption of regularity, provided both G and H are BCH-Lie groups (cf. G. 2002b).

Open Problem

Is every Lie group G modelled on a sequentially complete locally convcex space regular?

Can replace C^∞-maps with C^k-maps in the definition of regularity ("C^k-regularity", G. 2016); or with L^p-maps ("L^p-regularity", G. 2015, Nikitin 2021), if the modelling space is sequentially complete. Then C^∞-regularity equals regularity and (G. 2015–16)
L^1-reg $\Rightarrow$ L^p-reg $\Rightarrow$ L^∞-reg $\Rightarrow$ C^0-reg $\Rightarrow$ C^k-reg $\Rightarrow$ C^∞-reg

Facts. Banach–Lie groups, $\mathrm{Diff}(M)$ and $\mathrm{Diff}^\omega(M)$ are L^1-regular (G. 2015 resp. 2020); direct limit Lie groups $\lim\limits_{\rightarrow} G_n$ are L^1-regular (G. 2015); $C^n(M,G)$ is C^k-regular if so is G (G. 2016) and L^1-regular if G is a Banach–Lie group (G. 2015); unit groups $A^\times$ are C^0-regular if the cia A is sequentially complete and locally m-convex (or slightly more general) by G.–Neeb 2012; if A is locally m-convex and a Fréchet space, then $A^\times$ is L^1-regular (G. 2015).

In a nutshell: All Lie groups encountered in practice are regular!

Detailed analysis of regularity properties by Hanusch ('19 and '22). Notably, every C^0-semiregular Fréchet–Lie group is C^0-regular (Hanusch '19). If Evol exists, it is automatically smooth!

Remark. C^0-regularity implies the Trotter formula

$$\exp_G(x+y) = \lim_{n\to\infty} \big(\exp_G(x/n)\exp_G(y/n)\big)^n$$

and commutator formula (Hanusch 2020, which extends G. '15).

Remarks. (a) For regularity in the context of convenient differential calculus, see Kriegl-Michor 1997. This is an inequivalent setting of calculus; the smooth maps considered there need not be continuous (but they coincide with ours in the case of mappings between open subsets of Fréchet spaces).

(b) If a Lie group G acts smoothly on a C^∞-manifold M on the right (which may be infinite-dimensional), one can consider "fundamental" vector fields

$$v_\sharp\colon M \to TM, \quad m \mapsto \tfrac{d}{dt}\Big|_{t=0} m.\exp_G(tv)$$

for $v \in L(G)$. If G is regular (resp., L^1-regular) and $\eta \in C^\infty([0,1], L(G))$ (resp., $\eta \in L^1([0,1], L(G))$), then the ODE $\dot{y} = \eta(t)_\sharp(y(t))$ on M satisfies local and global existence and uniqueness of solutions (see G.–Neeb '23, G.–Hilgert '23).

(c) Regularity has also been established for Lie groups of Lie-group-valued mappings on $M := \mathbb{R}^m$ and for Lie groups of diffeomorphisms of $\mathbb{R}^m$ which are modelled on Schwartz spaces of rapidly decreasing smooth functions or other weighted function spaces (Walter '12, Nikitin '15).

§7 Difficulties in infinite dimensions

- Beyond Banach spaces, there are examples of initial value problems without solutions/with many solutions (see Milnor 82); no local existence/no local uniqueness!
- Beyond Banach spaces, no inverse function theorem in the ordinary form. (In Fréchet spaces, the Nash-Moser-Inverse Function Theorem is available with complicated hypotheses, see Hamiltonb '82. For maps to Banach spaces there still is an implicit function theorem, see G. 2006).

In Lie theory:

- Exponential function may not exist, need not be a local diffeomorphism at 0.
- A locally convex topological Lie algebra $\mathfrak{g}$ need not be the Lie algebra of a Lie group! Counterexamples for Banach–Lie algebras were first given by van Est and Korthagen '64, a simple example by Douady and Lazard '66.
- Closed subgroups need not be Lie subgroups

Three levels:

global level (Lie groups modelled on locally convex spaces)

local level (local Lie groups or germs of such)

infinitesimal level (locally convex topological Lie algebras)

Can always go down; in finite-dimensional case, can go up.[1] In the infinite-dimensional case, we can go up only under additional conditions, for each level !

For example, every Banach–Lie algebra $\mathfrak{g}$ gives rise to a local Banach–Lie group (via the Cambell-Hausdorff multiplication on a 0-neighbourhood in $\mathfrak{g} \times \mathfrak{g}$). But the latter need not arise from a global Banach–Lie group.

[1] Each finite-dimensional Lie algebra is the Lie algebra of some Lie group by Lie's Third Theorem.

For an infinite-dimensional complex Hilbert space $\mathcal{H}$, consider the unitary group $U(\mathcal{H})$, which is contractible by Kuiper's Theorem. Its Lie algebra is

$$\mathfrak{u}(\mathcal{H}) := i \operatorname{Herm}(\mathcal{H}).$$

As the elements of $\mathbb{R}\, i \operatorname{id}_{\mathcal{H}}$ commute with each operator, $\mathbb{R}\, i \operatorname{id}_{\mathcal{H}}$ is a central ideal of $\mathfrak{u}(\mathcal{H})$. As a consequence, $\mathbb{R}\, i \operatorname{id}_{\mathcal{H}} \times \mathbb{R}\, i \operatorname{id}_{\mathcal{H}}$ is a central ideal of $\mathfrak{u}(\mathcal{H}) \times \mathfrak{u}(\mathcal{H})$ and so is its vector subspace

$$\mathfrak{n} := \mathbb{R} i(\operatorname{id}_{\mathcal{H}}, \sqrt{2}\operatorname{id}_{\mathcal{H}}).$$

We claim that $\mathfrak{g} := (\mathfrak{u}(\mathcal{H}) \times \mathfrak{u}(\mathcal{H}))/\mathfrak{n}$ cannot be the Lie algebra of a Lie group. Otherwise, if $\mathfrak{g} = L(G)$, get a contradiction: The modelling space of G being being isomorphic to $\mathfrak{g}$ and hence a Banach space, G is a Banach–Lie group and thus BCH. Hence, there exists a real analytic group homomorphism

$$\alpha\colon U(\mathcal{H}) \times U(\mathcal{H}) \to G$$

such that $L(\alpha)$ is the quotient map $\beta\colon \mathfrak{u}(\mathcal{H}) \times \mathfrak{u}(\mathcal{H}) \to \mathfrak{g}$.

The exponential function exp of $U(\mathcal{H}) \times U(\mathcal{H})$ takes $\mathfrak{n}$ to a dense subgroup N of the 2-dimensional torus

$$T := \exp(i\mathbb{R}\,\mathrm{id}_{\mathcal{H}} \times i\mathbb{R}\,\mathrm{id}_{\mathcal{H}}) = \mathbb{S}_1\,\mathrm{id}_{\mathcal{H}} \times \mathbb{S}_1\,\mathrm{id}_{\mathcal{H}} \cong \mathbb{S}_1 \times \mathbb{S}_1,$$

a so-called "dense wind". By naturality of exp, we must have $N \subseteq \ker(\alpha)$. Since $\ker(\alpha)$ is closed, $T \subseteq \ker(\alpha)$ follows and hence $\mathbb{R}i\,\mathrm{id}_{\mathcal{H}} \times \mathbb{R}i\,\mathrm{id}_{\mathcal{H}} = L(T) \subseteq \ker(\beta)$, which contradicts the definition of β.

Remark. Świerczkowski '71 showed that non-integrability of a Banach–Lie algebra is always related to non-closedness of a relevant subgroup. Further examples of non-integrable Banach–Lie algebras can be found in G.–Neeb 2003.

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