Introduction to Infinite-Dimensional Lie Groups II

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Overview

Today:

- Definition of infinite-dimensional manifolds and Lie groups
- Examples:
  - Linear Lie groups $G \subseteq A^\times$
  - Mapping groups $C^n(M, G)$, notably $C^n(\mathbb{S}_1, G)$
  - Lie groups $\mathrm{Diff}(M)$ of smooth diffeomorphisms
  - Direct limit groups $G = \bigcup_{n\in\mathbb{N}} G_n$ with $G_1 \subseteq G_2 \subseteq \cdots$, e.g.

$$\mathrm{GL}_\infty(\mathbb{R}) = \bigcup_{n\in\mathbb{N}} \mathrm{GL}_n(\mathbb{R})$$

identify $A \in \mathrm{GL}_n(\mathbb{R})$ with $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R})$.

tomorrow!

§0 Repetition

E, F locally convex topological vector spaces, $U \subseteq E$ open

Definition (Andrée Bastiani, 1964)

A map $f\colon U \to F$ is called **continuously differentiable** (or C^1) if it is continuous, the directional derivatives

$$df(x,y) := (D_y f)(x) := \frac{d}{dt}\Big|_{t=0} f(x+ty) = \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$$

exist for all $x \in U$ and $y \in E$, and $df\colon U \times E \to F$, $(x,y) \mapsto df(x,y)$ is continuous.

If f is C^1 and $df\colon U \times E \to F$ is C^n, call f a C^{n+1}-map.

f is C^n if and only if f is continuous and the iterated directional derivatives $d^j f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$ exist for all $j \in \mathbb{N}$ such that $j \leq n$, all $x \in U$ and $y_1, \ldots, y_j \in E$, and $d^j f\colon U \times E^j \to F$ is continuous.

$d^j f(x, \cdot)\colon E^j \to F$ is a continuous and symmetric j-linear map

§1 Analytic mappings

If E and F are locally convex spaces over $\mathbb{C}$, replacing real directional derivatives with

$$\frac{d}{dz}\Big|_{z=0} f(x+zy)$$

get $C^1_{\mathbb{C}}$-maps $f\colon U \to F$ on $U \subseteq E$ and $C^n_{\mathbb{C}}$ maps.

Fact

Let E and F be complex locally convex spaces, and $U \subseteq E$ be an open subset. For a map $f\colon U \to F$, the following are equivalent:

(a) f is $C^\infty_{\mathbb{C}}$;

(b) f is C^∞ and $df(x,\cdot)\colon E \to F$ is $\mathbb{C}$-linear for all $x \in U$;

(c) f is **complex analytic**, i.e., f is continuous and each $x \in U$ has an open neighbourhood $V \subseteq U$ such that

$$f(y) = f(x) + \sum_{n=1}^{\infty} \beta_n(y-x,\ldots,y-x) \quad \text{for all } y \in V$$

with pointwise convergence, for continuous complex n-linear maps $\beta_n\colon E^n \to F$.

If F is sequentially complete, then also the following condition is equivalent:

(d) f is $C^1_{\mathbb{C}}$.

Let E and F be real locally convex spaces and $U \subseteq E$ be open. A map $f\colon U \to F$ is called **real analytic** if it has a complex analytic extension $\widetilde{f}\colon \widetilde{U} \to F \oplus iF$ for some open subset $\widetilde{U} \subseteq E \oplus iE$ with $U \subseteq \widetilde{U}$, i.e., $f = \widetilde{f}|_U$.

Compositions of complex analytic maps are complex analytic.
Compositions of real analytic maps are real analytic.

Every complex analytic map is real analytic.

Every real analytic map is C^∞.

§2 Manifolds modelled on a locally convex space

Let E be a locally convex topological vector space and M be a topological space which is Hausdorff (all $x \neq y$ have disjoint open neighbourhoods). An ***E*-chart** for M is a map

$$\phi\colon U_\phi \to V_\phi$$

from an open subset $U_\phi \subseteq M$ onto an open subset $V_\phi \subseteq E$ which is a **homeomorphism** (i.e., ϕ is invertible and both ϕ and ϕ^{-1} are continuous).

Given $n \in \mathbb{N}$ call E-charts $\phi\colon U_\phi \to V_\phi$ and $\psi\colon U_\psi \to V_\psi$ **C^n-compatible** if both

$$\psi \circ \phi^{-1}\colon \phi(U_\phi \cap U_\psi) \to E, \quad x \mapsto \psi(\phi^{-1}(x))$$

and $\phi \circ \psi^{-1}$ are C^n-maps. A set $\mathcal{A}$ of C^n-compatible E-charts $\phi\colon U_\phi \to V_\phi$ for M is called a **C^n-atlas** for M if

$$\bigcup_{\phi \in \mathcal{A}} U_\phi = M\,.$$

Definition

A ***C^n*-manifold** modelled on E is a Hausdorff topological space M, together with a C^n-atlas $\mathcal{A}$ of E-charts which is maximal.

Every C^n-atlas is contained in a unique maximal C^n-atlas (all E-charts C^n-compatible with the given ones).

If $(M, \mathcal{A})$ is a C^n-manifold modelled on E, then the $\phi \in \mathcal{A}$ shall simply be called "charts for M." The inverses $\phi^{-1}\colon E \supseteq V_\phi \to U_\phi \subseteq M$ are called **local parametrizations** of M.

If M is modelled on a Banach space, call M a **Banach manifold**. Call M a Fréchet manifold if it is modelled on a **Fréchet space** E (E is sequentially complete & topology comes from a countable set of seminorms).

Definition

Let M and N be C^n-manifolds modelled on locally convex spaces. A map $f\colon M \to N$ is called a **C^n-map** if f is continuous and $\phi \circ f \circ \psi^{-1}$ is C^n for each chart ϕ of N and each chart ψ of M.

It's enough to check this for ϕ and ψ in a subatlas.

Example. If E is a locally convex space, then every open subset $U \subseteq E$ is a smooth manifold modelled on E, using the maximal C^∞-atlas containing the E-chart $\mathrm{id}_U \colon U \to U$, $x \mapsto x$.

Example. If $(M_1, \mathcal{A}_1)$ and $(M_2, \mathcal{A}_2)$ are C^n-manifolds modelled on locally convex spaces E_1 and E_2, respectively, endow $M := M_1 \times M_2$ with the product topology (unions of products $U_1 \times U_2$ of open subsets $U_1 \subseteq M_1$ and $U_2 \subseteq M_2$ are open) and give it the maximal C^n-atlas containing the $E_1 \times E_2$-charts

$$\phi \times \psi \colon U_\phi \times U_\psi \to V_\phi \times V_\psi \subseteq E_1 \times E_2, \ \ (x_1, x_2) \mapsto (\phi(x_1), \psi(x_2))$$

for $\phi \in \mathcal{A}_1$, $\psi \in \mathcal{A}_2$.

This product manifold structure on $M_1 \times M_2$ turns the projections

$$\mathrm{pr}_j \colon M_1 \times M_2 \to M_j, \quad (x_1, x_2) \mapsto x_j$$

into C^n-maps for $j \in \{1, 2\}$. For each C^n-manifold N, a map $f = (f_1, f_2) \colon N \to M_1 \times M_2$ is C^n if and only if both components $f_1 \colon N \to M_1$ and $f_2 \colon N \to M_2$ are C^n.

Definition

A **Lie group** is a group G, endowed with a smooth manifold structure modelled on a locally convex topological vector space E, such that the group multiplication

$$\mu_G \colon G \times G \to G, \quad (g,h) \mapsto gh$$

and the inversion map

$$\eta_G \colon G \to G, \quad g \mapsto g^{-1}$$

are smooth.

Example. For every continuous algebra A, the open unit group $A^\times \subseteq A$ is a Lie group.

Remark

For $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{R}$, can define $\mathbb{K}$-analytic manifolds modelled on a locally convex topological $\mathbb{K}$-vector space and $\mathbb{K}$-analytic maps between such as before, replacing C^n-maps with $\mathbb{K}$-analytic maps.

Likewise, replacing C^∞-maps with $\mathbb{K}$-analytic maps in the definition of a Lie group get concept of a $\mathbb{K}$-**analytic Lie group**.

Example. For each complex continuous inverse algebra A, the unit group $A^\times$ is a complex analytic Lie group.

In fact, the algebra multiplication $\beta\colon A \times A \to A$ is complex bilinear and thus $\mathbb{C}$-analytic, as

$$d\beta((x_1, x_2), (y_1, y_2)) = \beta(y_1, x_2) + \beta(x_1, y_2)$$

is $\mathbb{C}$-linear in (y_1, y_2). Likewise, the smooth inversion map $\eta\colon A^\times \to A$ is $\mathbb{C}$-analytic as $d\eta(x, y) = -x^{-1}yx^{-1}$ is $\mathbb{C}$-linear in y.

Example. For each real continuous inverse algebra A, its open unit group $A^\times \subseteq A$ is a real analytic Lie group.

For the proof, one shows that $A_{\mathbb{C}} = A \oplus iA$ is a complex cia. The group operations of $(A_{\mathbb{C}})^\times$ are $\mathbb{C}$-analytic and extend those of $A^\times$, whence the latter are real analytic.

Definition

Let M be a C^n-manifold modelled on a locally convex space E. Let $F \subseteq E$ be a closed vector subspace. A subset $N \subseteq M$ is called a **submanifold** of M modelled on F if for each $x \in N$, there exists a chart $\phi\colon U_\phi \to V_\phi \subseteq E$ of M with $x \in U_\phi$ which is **adapted to** N in the sense that

$$\phi(N \cap U_\phi) = F \cap V_\phi.$$

The restrictions $\phi_N\colon N \cap U_\phi \to F \cap V_\phi$, $y \mapsto \phi(y)$ of charts ϕ adapted to N are called **submanifold charts** for N; they form a C^n-atlas of F-charts for N if we give N the topology induced by M. We endow N with the corresponding maximal C^n-atlas. Then the following holds;

Facts.

(a) The inclusion map $j\colon N \to M$, $x \mapsto x$ is C^n.

(b) For every C^n-manifold L, a map $f\colon L \to N$ is C^n if and only if the map $j \circ f\colon L \to M$ is C^n.

Fact

If G is a Lie group and $H \subseteq G$ a subgroup which is a submanifold, then the smooth manifold structure on H as a submanifold makes H a Lie group.

For example, consider the smooth inversion map $\eta_G \colon G \to G$ and the inversion map $\eta_H \colon H \to H$. As the inclusion map $j \colon H \to G$ is smooth, also the map

$$\eta_G \circ j \colon H \to G, \quad h \mapsto h^{-1}$$

is smooth, which coincides with $j \circ \eta_H$. Hence η_H is smooth, by the preceding fact (b).

Subgroups $H \subseteq G$ which are submanifolds are called **Lie subgroups**.

A subgroup $H \subseteq G$ is a submanifold modelled on F if there is a chart $\phi \colon U \to V$ of G with $e \in U$ and $\phi(H \cap U) = F \cap V$.
[For each $h \in H$, the map $hU \to V$, $x \mapsto \phi(h^{-1}x)$ then is a chart of G adapted to H, with h in its domain.]

Definition

Lie subgroups $H \subseteq A^\times$ for a continuous inverse algebra A are called **linear Lie groups**.

Example

For each complex Hilbert space $\mathcal{H}$, the unitary group $U(\mathcal{H}) := \{T \in B(\mathcal{H})\colon\, TT^* = T^*T = \mathrm{id}_{\mathcal{H}}\}$ is a Lie subgroup of $\mathrm{GL}(\mathcal{H}) := B(\mathcal{H})^\times$ and hence a linear Lie group.

In fact, the exponential map

$$\exp\colon B(\mathcal{H}) \to B(\mathcal{H}), \quad T \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} T^n$$

is complex analytic and $\exp'(0) = \mathrm{id}_{B(\mathcal{H})}$. By the Inverse Function Theorem, for small $\varepsilon > 0$ the image $U := \exp(V)$ is open for the ball

$$V := \{T \in B(\mathcal{H})\colon \|T\|_{\mathrm{op}} < \varepsilon\}$$

and $\theta := \exp|_V\colon V \to \exp(V) \subseteq \mathrm{GL}(\mathcal{H})$ is a complex analytic diffeomorphism and hence a local parametrization of $\mathrm{GL}(\mathcal{H})$ (i.e., $\phi := \theta^{-1}\colon U \to V$ is a chart with $U := \exp(V)$).

For each $T \in V$, we have

$$\theta(T)^{-1} = \exp(T)^{-1} = \exp(-T) = \theta(-T)$$

and

$$\theta(T)^* = \exp(T)^* = \exp(T^*) = \theta(T^*).$$

As θ is injective, we deduce that $\theta(T)^{-1} = \theta(T)^*$ if and only if $T^* = -T$, i.e., if and only if T is skew-hermitian. Let $\mathrm{Herm}(\mathcal{H}) := \{T \in B(\mathcal{H})\colon T = T^*\}$ be the set of hermitian bounded linear operators, which is a closed real vector subspace of $B(\mathcal{H})$. By the preceding,

$$\theta(i\,\mathrm{Herm}(\mathcal{H}) \cap V) = U(\mathcal{H}) \cap U$$

and hence $\phi(U(\mathcal{H}) \cap U) = i\,\mathrm{Herm}(\mathcal{H}) \cap V$, entailing that $U(\mathcal{H})$ is a submanifold (and hence a Lie subgroup) of $\mathrm{GL}(\mathcal{H})$ modelled on the closed real vector subspace $i\,\mathrm{Herm}(\mathcal{H}) \subseteq B(\mathcal{H})$ of skew-hermitian operators.

Remark. We mention that the unitary group $U(\mathcal{H})$ is **contractible** for each infinite-dimensional complex Hilbert space $\mathcal{H}$, by Kuiper's Theorem. That is, there exist a continuous map

$$F\colon [0,1] \times U(\mathcal{H}) \to U(\mathcal{H})$$

such that $F(0,T) = T$ for all $T \in U(H)$ and $F(1,\cdot)$ is a constant map.

By contrast, $U(\mathbb{C}^n)$ is not contractible for $n \in \mathbb{N}$ and not even simply connected. For example,

$$U(\mathbb{C}^1) \cong \mathbb{S}_1$$

with $\pi_1(\mathbb{S}_1) \cong \mathbb{Z}$.

§4 Local construction of Lie groups

Let G be a group and $P \subseteq G$ be a subset with $e \in P$ which is endowed with a smooth manifold structure modelled on a locally convex space E. Let $Q \subseteq P$ be an open subset with $e \in Q$ and $Q = Q^{-1}$ such that $QQ \subseteq P$.

Fact.

Assume that

(a) The restriction of the group multiplication to a map $Q \times Q \to P$ is smooth;

(b) The restriction of the inversion map to a map $Q \to Q$ is smooth;

(c) For each $g \in G$, there exists an open e-neighbourhood $W \subseteq P$ such that $gWg^{-1} \subseteq P$ and the map $W \to P$, $x \mapsto gxg^{-1}$ is smooth.

Then there is a unique smooth manifold structure on G which is modelled on E, turns G into a Lie group and turns Q into an open submanifold of G.

For chart a chart $\phi\colon U_\phi \to V_\phi$ of Q with $e \in U_\phi$, define

$$\phi_g\colon gU_\phi \to V_\phi, \quad x \mapsto \phi(g^{-1}x)\,.$$

One verifies that $\{\phi_g \colon g \in G\}$ is a C^∞-atlas and that the corresponding smooth manifold structure on G has the asserted properties.

Recall that the supremum norm $\|\cdot\|_\infty$ makes $C([0,1],\mathbb{R})$ a Banach space. Likewise for $C([0,1],\mathbb{R}^n)$, using $\max_{t\in[0,1]}\|\gamma(t)\|$ for a given norm $\|\cdot\|$ on $\mathbb{R}^n$.

Fact

For every continuous map $f\colon \mathbb{R}^n\to\mathbb{R}^m$, the map

$$f_* := C([0,1],f)\colon C([0,1],\mathbb{R}^n)\to C([0,1],\mathbb{R}^m),\quad \gamma\mapsto f\circ\gamma$$

is continuous.

To see continuity at γ, note that $\gamma([0,1])$ is compact, hence contained in a closed ball $\overline{B}_r(0) := \{y\in\mathbb{R}^n\colon \|y\|\leq r\}$ for some $r>0$. On the compact ball $\overline{B}_{2r}(0)$, the continuous map f is uniformly continuous: Given $\varepsilon>0$, there exists $\delta>0$ such that

$$\|f(y)-f(x)\|\leq\varepsilon\quad\text{for all } x,y\in\overline{B}_{2r}(0) \text{ such that } \|y-x\|\leq\delta. \quad (*)$$

We may assume that $\delta\leq r$. For each η in the open ball

$$\{\eta \in C([0,],\mathbb{R}^n)\colon \|\eta-\gamma\|_\infty < \delta\}$$
$$\subseteq \{\theta \in C([0,1],\mathbb{R}^n)\colon \|\theta\|_\infty < 2r\} = C([0,1],B_{2r}(0)),$$

we then have

$$\|\eta(t)-\gamma(t)\| < \delta$$

for each $t \in [0,1]$ and hence

$$\|f(\eta(t)) - f(\gamma(t))\| \leq \varepsilon$$

by $(*)$, as $\gamma(t),\eta(t) \in \overline{B}_{2r}(0)$. Thus

$$\|f\circ\eta - f\circ\gamma\|_\infty \leq \varepsilon\,.$$

If $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is C^1, then $f_* := C([0,1],f)$ is C^1, with

$d(f_*) = (df)_*\colon C([0,1],\mathbb{R}^n\times\mathbb{R}^n) \to C([0,1],\mathbb{R}^m)$, $(\gamma,\eta)\mapsto df\circ(\gamma,\eta)$.

For the proof, fix $\gamma,\eta \in C([0,1],\mathbb{R}^n)$ and consider for $t \neq 0$ the difference quotient

$$\Delta_t := \frac{f_*(\gamma + t\eta) - f_*(\gamma)}{t} \in C([0,1],\mathbb{R}^m)$$

and the map

$$h\colon \mathbb{R} \to C([0,1],\mathbb{R}^m), \quad t \mapsto \int_0^1 (df)_*(\gamma + st\eta, \eta)\, ds,$$

which is continuous as $(df)_*(\gamma + st\eta, \eta)$ is continuous in (s,t). For each $x \in [0,1]$, the point evaluation

$$\varepsilon_x\colon C([0,1],\mathbb{R}^m) \to \mathbb{R}^m, \quad \theta \mapsto \theta(x)$$

is continuous and linear. As it commutes with the weak integral, we get

$$\varepsilon_x(h(t)) = \int_0^1 df(\gamma(x) + st\eta(x), \eta(x))\, ds = \frac{f(\gamma(x) + t\eta(x)) - f(\gamma(x))}{t} = \Delta_t(x)$$

using the Mean Value Theorem. Thus $\Delta_t = h(t) \to h(0) = df \circ (\gamma, \eta)$ as $t \to 0$.

If E is a locally convex space and $K = [0,1]$, $K = \mathbb{S}_1$ or K a compact topological space, make the space $C(K, E)$ of continuous maps $\gamma\colon K \to E$ a locally convex topological vector space using the seminorms

$$\|\gamma\|_q := \sup_{x \in K} q(\gamma(x))$$

for continuous seminorms $q\colon E \to [0, \infty[$. Like the preceding special cases, one shows:

Fact

If E and F are locally convex spaces, $U \subseteq E$ is an open subset and $f\colon U \to F$ is a C^n-map with $n \in \mathbb{N}_0 \cup \{\infty\}$, then $C(K, U)$ is open in $C(K, E)$ and the map

$$f_* := C(K, f)\colon C(K, U) \to C(K, F), \quad \gamma \mapsto f \circ \gamma$$

is C^n.

This implies:

Fact

For every Lie group G modelled on a locally convex space E, the group $C(K,G)$ of all continuous maps $\gamma\colon K\to G$ is a Lie group modelled on $C(K,E)$, with $(\gamma\eta)(x):=\gamma(x)\eta(x)$.

Let $\phi\colon U\to V\subseteq E$ be a chart of G with $e\in U$. Give $P:=C(K,U)$ the smooth manifold structure turning the bijection

$$\phi_*\colon C(K,U)\to C(K,V)\subseteq C(K,E),\quad \gamma\mapsto\phi\circ\gamma$$

into a C^∞-diffeomorphism (smooth with smooth inverse). Let $W\subseteq U$ be an open subset with $WW\subseteq U$ and $W=W^{-1}$. Then $\phi(W)$ is open in V and $Q:=C(K,W)$ is open in P, as it corresponds to the open subset $C(K,\phi(W))$ of $C(K,V)$. The inversion map $\eta_{C(K,G)}|_Q$ on Q corresponds to the map

$$h\colon C(K,\phi(W))\to C(K,\phi(W)),\quad \gamma\mapsto(\phi\circ\eta_G^{-1}\circ\phi^{-1})\circ\gamma,$$

which is smooth by the preceding fact. Hence $\eta_{C(K,G)}|_Q=(\phi_*)^{-1}\circ h\circ\phi_*|_Q$ is smooth. Similarly, Conditions (b) and (c) of the local description of Lie group structures are satisfied.

Let M be an m-dimensional compact C^∞-manifold, E be a locally convex topological vector space and $n \in \mathbb{N}_0 \cup \{\infty\}$. For a chart $\phi\colon U_\phi \to V_\phi$ of M, $j \in \mathbb{N}_0$ with $j \leq n$, a compact subset K of $V_\phi \times (\mathbb{R}^m)^j$ and a continuous seminorm q on E, a seminorm

$$\|\cdot\|_{\phi,j,q,K}\colon C^n(M,E) \to [0,\infty[, \quad \gamma \mapsto \|q \circ d^j(f \circ \phi^{-1})|_K\|_\infty$$

is obtained. Make $C^n(M,E)$ a locally convex topological vector space using these seminorms (the resulting topology is called the **compact-open C^n-topology**).

Fact

If also F is a locally convex topological vector space, $k \in \mathbb{N}_0 \cup \{\infty\}$ and $f\colon U \to F$ is a C^{n+k}-map on an open subset $U \subseteq E$, then $C^n(M,U)$ is open in $C^n(M,E)$ and the map

$$f_* := C^n(M,f)\colon C^n(M,U) \to C^n(M,F), \quad \gamma \mapsto f \circ \gamma$$

is C^k.

As before, this entails:

Fact

For every Lie group G modelled on a locally convex space E, the group $C^n(M,G)$ of all C^n-maps $\gamma\colon M \to G$ is a Lie group modelled on $C^n(M,E)$.

We can use $P = C^n(M,U)$ with the smooth manifold structure making $\phi_*\colon C^n(M,U) \to C^n(M,V) \subseteq C^n(M,E)$, $\gamma \mapsto \phi \circ \gamma$ a C^∞-diffeomorphism and $Q := C^n(M,W)$ as in the preceding proof.

Bibliography

- Bastiani, A., *Applications différentiables et variétés différentiables de dimension infinie*, J. Anal. Math. **13** (1964), 1–114.
- Bochnak, J. and J. Siciak, *Analytic functions in topological vector spaces*, Stud. Math. **39** (1971), 77–112.
- Bourbaki, N., "Lie Groups and Lie Algebras, Chapters 1–3," Springer, Berlin, 1989.
- Eells, J. jun., *A setting for global analysis*, Bull. Amer. Math. Soc. **72** (1966), 751–807.
- Glöckner, H., *Infinite-dimensional Lie groups without completeness restrictions*, pp. 43–59 in: Strasburger, A. et al. (eds.), "Geometry and Analysis on Finite- and Infinite-Dimensional Lie Groups," Banach Center Publications **55**, Warsaw, 2002.
- Glöckner, H., *Algebras whose groups of units are Lie groups*, Stud. Math. **153** (2002), 147–177.
- Glöckner, H., *Lie group structures on quotient groups and universal complexifications for infinite-dimensional Lie groups*, J. Funct. Anal. **194** (2002), 347–409.
- Glöckner, H. and K.-H. Neeb, "Infinite-Dimensional Lie Groups," book in preparation, 2023.

- Hamilton, R. S., *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. (1982), 65–222.
- de la Harpe, P., "Classical Banach–Lie Algebras and Banach–Lie Groups of Operators in Hilbert Space," Springer, Berlin, 1972.
- Kuiper, N. H., *The homotopy type of the unitary group of Hilbert space*, Topology **3** (1965), 19–30.
- Lang, S. "Fundamentals of Differential Geometry," Springer, New York, 2001.
- Maissen, B., *Lie-Gruppen mit Banachräumen als Parameterräume*, Acta Math. **108** (1962), 229–270.
- Michor, P. W., "Manifolds of Differentiable Mappings," Shiva Publishing, Orpington, 1980.
- Milnor, J., *Remarks on infinite-dimensional Lie groups*, pp. 1007–1057 in: B. S. DeWitt and R. Stora (eds.), "Relativité, groupes et topologie II," North-Holland, Amsterdam, 1984.
- Neeb, K.-H., *Towards a Lie theory of locally convex groups*, Jpn. J. Math. **1** (2006), 291–468.
- Pressley, A. and G. Segal, "Loop Groups," Clarendon Press, Oxford, 1988.