Introduction to Infinite-Dimensional Lie Groups

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Overview

Today: calculus in locally convex spaces

Wednesday and Thursday: Examples and theory of infinite-dimensional Lie groups

It's not sufficient for our purposes to consider normed spaces $(E, \|\cdot\|)$ or Banach spaces. Rather, consider locally convex topological vector spaces E whose topology comes from a set Γ of seminorms

$$q\colon E \to [0,\infty[, \quad x \mapsto q(x)\,.$$

Thus $q(tx) = |t|q(x)$ for all $t \in \mathbb{R}$ and $x \in E$; and

$$q(x+y) \leq q(x) + q(y) \quad \text{for all } x, y \in E.$$

For each $x \in E \setminus \{0\}$, we have $q(x) > 0$ for some Γ. We can assume that for all $p, q \in \Gamma$, the seminorm

$$x \mapsto \max\{p(x), q(x)\}$$

is in Γ.

A subset $U \subseteq E$ is open if, for each $x \in U$, it contains the ball

$$B^q_\varepsilon(x) := \{y \in E\colon q(y-x) < \varepsilon\}$$

for some $q \in \Gamma$ and some $\varepsilon > 0$.

Example. Every normed space is a locally convex space, notably

(a) The set $C(K, \mathbb{R})$ of continuous real-valued functions with the supremum norm $\|\cdot\|_\infty$;

(b) For $k \in \mathbb{N}_0$ the space $C^k([0,1], \mathbb{R})$ of C^k-functions $f\colon [0,1] \to \mathbb{R}$,

$$\|f\|_{C^k} := \max_{j=0,\ldots,k} \|f^{(j)}\|_\infty .$$

Example. The space $C^\infty([0,1], \mathbb{R})$ of smooth functions becomes a locally convex space if we use the seminorms $\|\cdot\|_{C^k}$ for $k \in \mathbb{N}_0$.

Definition

A sequence $(x_n)_{n\in\mathbb{N}}$ in a locally convex space is called a **Cauchy sequence** if, for each 0-neighbourhood $U \subseteq E$, there exists $N \in \mathbb{N}$ such that

$$x_m - x_n \in U \quad \text{for all } n, m \geq N.$$

If every Cauchy sequence in E is convergent, then E is called **sequentially complete**.

For example, $C^k([0,1],\mathbb{R})$ is sequentially complete for all $k \in \mathbb{N}_0$ and also for $k = \infty$.

Consider $a < b$ and a continuous function $\gamma\colon [a,b] \to E$.

Definition

An element $x \in E$ is called a *weak integral* for γ if

$$\lambda(x) = \int_a^b \lambda(\gamma(t))\, dt$$

for each continuous linear functional $\lambda\colon E \to \mathbb{R}$.

If it exists, the weak integral is unique, as E' separates points on E be the Hahn-Banach Theorem. Write $\int_a^b \gamma(t)\, dt := x$; thus

$$\lambda\left(\int_a^b \gamma(t)\, dt\right) = \int_a^b \lambda(\gamma(t))\, dt$$

for all $\lambda \in E'$.

Fact

If E is sequentially complete, then the weak integral $\int_a^b \gamma(t)\,dt$ exists in E for each continuous function $\gamma\colon [a,b] \to E$.

For the proof, consider the Riemann sum

$$S(\gamma, Z) := \sum_{j=1}^{k} (t_j - t_{j-1})\,\gamma(t_j)$$

for a subdivision $Z = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of $[a,b]$, and

$$\Delta(Z) := \max_{j=1,\ldots,k} (t_j - t_{j-1}).$$

Pick a sequence $(Z_n)_{n\in\mathbb{N}}$ of subdivisions with $\Delta(Z_n) \to 0$. Since γ is uniformly continuous, see that $(S(f, Z_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, hence convergent to some $x \in E$. Applying $\lambda \in E'$, get

$$\lambda(S(\gamma, Z_n)) = \sum_{j=1}^{k} (t_j - t_{j-1})\lambda(\gamma(t_j)) = S(\lambda\circ\gamma, Z_n) \to \int_a^b (\lambda(\gamma(t))\,dt$$

and thus $\lambda(x) = \lim_{n\to\infty} \lambda(S(f, Z_n)) = \int_a^b \lambda(\gamma(t))\,dt$.

Let E be a locally convex space.

Definition

A continuous map $\gamma\colon [a,b] \to E$ is called a **C^1-curve** if

$$\gamma'(t) := \lim_{s\to t} \frac{\gamma(s)-\gamma(t)}{s-t}$$

exists in E for each $t \in [a,b]$ and $\gamma'\colon [a,b] \to E$ is continuous.

Fundamental Theorem

If $\gamma\colon [a,b] \to E$ is a C^1-curve, then

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)\, dt\,.$$

Notably, the weak integral exists.

Proof. For each continuous linear functional $\lambda\colon E \to \mathbb{R}$, the composition $\lambda \circ \gamma \colon [a, b] \to \mathbb{R}$ is continuous and for $t \in [a, b]$, we have

$$\frac{\lambda(\gamma(s)) - \lambda(\gamma(t))}{s - t} = \lambda\left(\frac{\gamma(s) - \gamma(t)}{s - t}\right) \to \lambda(\gamma'(t)) \quad (*)$$

as $s \to t$, whence $\lambda \circ \gamma \colon [a, b] \to \mathbb{R}$ is a C^1-function. Thus

$$\lambda(\gamma(b) - \gamma(a)) = \lambda(\gamma(b)) - \lambda(\gamma(a)) = \int_a^b (\lambda \circ \gamma)'(t)\, dt = \int_a^b \lambda(\gamma'(t))\, dt,$$

using the Fundamental Theorem for the real-valued function $\lambda \circ \gamma$ and then $(*)$. Hence $\gamma(b) - \gamma(a)$ is the weak integral of γ'.

§3 C^1-maps between locally convex spaces

E, F	locally convex topological vector spaces
$U \subseteq E$	open

Definition (Andrée Bastiani, 1964)

A map $f\colon U \to F$ is called **continuously differentiable** (or C^1) if it is continuous, the directional derivatives

$$df(x,y) := (D_y f)(x) := \frac{d}{dt}\Big|_{t=0} f(x+ty) = \lim_{t\to 0} \frac{f(x+ty)-f(x)}{t}$$

exist for all $x \in U$ and $y \in E$, and $df\colon U \times E \to F$, $(x,y) \mapsto df(x,y)$ is continuous.

Example. Every constant map $f\colon E \to F$ is C^1 with $df(x,y) = 0$.

Example. Every continuous linear map $\alpha\colon E \to F$ is C^1, with $d\alpha(x,y) = \alpha(y)$ for all $x, y \in E$.

In fact, $\alpha(x+ty) = \alpha(x) + t\alpha(y)$ implies that

$$\frac{\alpha(x+ty) - \alpha(x)}{t} = \alpha(y) \to \alpha(y)$$

as $t \to 0$.

Example. Every continuous bilinear map $\beta\colon E_1 \times E_2 \to F$ is C^1, with

$$d\beta((x_1, x_2), (y_1, y_2)) = \beta(y_1, x_2) + \beta(x_1, y_2)$$

for all $x_1, y_1 \in E_1$ and $x_2, y_2 \in E_2$. In fact,

$$\beta(x_1+ty_1, x_2+ty_2) = \beta(x_1, x_2)+t\beta(y_1, x_2)+t\beta(x_1, y_2)+t^2\beta(y_1, y_2)$$

implies that

$$\frac{\beta(x_1+ty_1, x_2+ty_2) - \beta(x_1, x_2)}{t} = \beta(y_1, x_2)+\beta(x_1, y_2)+t\beta(y_1, y_2)$$

converges to $\beta(y_1, x_2) + \beta(x_1, y_2)$ as $t \to 0$.

Consider a C^1-map $f\colon E \supseteq U \to E$ as before.

Mean Value Theorem

Let $x, y \in U$. If $x + t(y-x) \in U$ for all $t \in [0,1]$, then

$$f(y) - f(x) = \int_0^1 df(x + t(y-x), y-x)\, dt\,.$$

Proof. Apply the Fundamental Theorem to $\gamma\colon [0,1] \to F$, $\gamma(t) := f(x + t(y-x))$ with $\gamma'(t) = df(x + t(y-x), y-x)$. $\square$

Define $U^{[1]} := \{(x,y,t) \in U \times E \times \mathbb{R} \colon x + ty \in U\}$

Lemma

If $f \colon U \to F$ is C^1, then the map

$$f^{[1]} \colon U^{[1]} \to F, \quad (x,y,t) \mapsto \begin{cases} \frac{f(x+ty)-f(x)}{t} & \text{if } t \neq 0; \\ df(x,y) & \text{if } t = 0 \end{cases}$$

is continuous.

On the open set $t \neq 0$, this follows from the continuity of f. For (x,y,t) close to $(x_0, y_0, 0)$,

$$f^{[1]}(x,y,t) = \int_0^1 df(x + sty, y)\, ds$$

by the Mean Value Theorem; integrals depend continuously on parameters (x,y,t).

Consider a C^1-map $f\colon E \supseteq U \to F$ as before and a C^1-map $g\colon V \to Y$ to a locally convex space Y, defined on an open subet $V \subseteq F$ such that $f(U) \subseteq V$.

Chain Rule

The composition $g \circ f\colon U \to Y$, $x \mapsto g(f(x))$ is C^1 with

$$d(g \circ f)(x, y) = dg(f(x), df(x, y)) \quad \text{for all } (x, y) \in U \times E.$$

Proof. We have

$$\frac{g(f(x+ty)) - g(f(x))}{t} = \frac{g\left(f(x) + t\frac{f(x+ty)-f(x)}{t}\right) - f(x)}{t} = g^{[1]}(f(x), f^{[1]}(x, y, t), t)$$

for $t \neq 0$, which converges to

$$g^{[1]}(f(x), f^{[1]}(x, y, 0), 0) = dg(f(x), df(x, y))$$

as $t \to 0$.

Remark

If $f\colon E \supseteq U \to F$ is C^1, then the continuous map $f'(x) := df(x,\cdot)\colon E \to F$ is linear for each $x \in U$.

We have $df(x,0y) = df(x,0) = 0 = 0df(x,y)$ and for $t \neq 0$

$$df(x,ty) = \lim_{s\to 0} \frac{f(x+sty)-f(x)}{s} = t\lim_{s\to 0} \frac{f(x+sty)-f(x)}{st} = tdf(x,y).$$

For $x \in U$ and $y_1, y_2 \in E$,

$$\begin{aligned} \frac{f(x+ty_1+ty_2)-f(x)}{t} &= \frac{f(x+ty_1+ty_2)-f(x+ty_1)}{t} \\ &\quad + \frac{f(x+ty_1)-f(x)}{t} \\ &= f^{[1]}(x+ty_1, y_2, t) + f^{[1]}(x, y_1, t) \end{aligned}$$

converges both to $df(x, y_1+y_2)$ and $f^{[1]}(x,y_2,0) + f^{[1]}(x,y_1,0) = df(x,y_1) + df(x,y_2)$ as $t \to 0$. The limits must coincide.

Fact

A map $f\colon E \supseteq U \to \prod_{j\in J} F_j =: F$ to a product of locally convex spaces with components $f_j\colon U \to F_j$ is C^1 if and only if each f_j is C^1. In this case,

$$df(x,y) = (df_j(x,y))_{j\in J} \quad \text{for all } x \in U \text{ and } y \in E. \qquad (1)$$

Proof. The projection $\mathrm{pr}_j\colon F \to F_j$, $(y_i)_{i\in J} \mapsto y_j$ onto the jth component is continuous and linear. Hence, if f is C^1, then also $f_j = \mathrm{pr}_j \circ f$ is C^1 and
$d(f_j)(x,y) = d(\mathrm{pr}_j)(f(x), df(x,y)) = \mathrm{pr}_j(df(x,y))$, whence (1) holds.

If all components are C^1, then

$$\frac{f(x+ty)-f(x)}{t} = \left(\frac{f_j(x+ty)-f_j(x)}{t}\right)_{j\in J}$$

converges to $(df_j(x,y))_{j\in J}$ as $t\to 0$ since this holds componen-wise.

Fact

Let $f\colon E \supseteq U \to F$ be a map and $F_0 \subseteq F$ be a closed vector subspace such that $f(U) \subseteq F_0$. Then f is C^1 if and only if $f|^{F_0}\colon U \to F_0$ is so, and $df(x,y) = d(f|^{F_0})(x,y)$ for all $(x,y) \in U \times E$ in this case.

Proof. The inclusion map $j\colon F_0 \to F$ is continuous and linear. Hence, if $f|^{F_0}$ is C^1, then also $f = j \circ f|^{F_0}$, with

$$df(x,y) = dj(f|^{F_0}(x), d(f|^{F_0})(x,y)) = j(d(f|^{F_0})(x,y)) = d(f|^{F_0})(x,y).$$

If f is C^1, then $f|^{F_0}$ is continuous and

$$\frac{f(x+ty) - f(x)}{t} \to df(x,y)$$

in F as $t \to 0$. Note that the difference quotients are in F_0; since F_0 is closed, the limit $df(x,y)$ is in F_0 as well and the difference quotients, which coincide with those of $f|^{F_0}$, converge to $df(x,y)$ also in F_0 using the induced topology.

The preceding two facts concerning mappings into products and mappings into closed vector subspaces hold just as well with C^k-maps in place of C^1-maps, and we shall not restate them. Likewise the Chain Rule (compositions of composable C^k-maps are C^k). Higher order differentiability is defined as follows.

Let E and F be locally convex spaces and $U \subseteq E$ be an open subset. Let $k \in \mathbb{N}$.

Definition

A map $f\colon U \to F$ is called C^{k+1} if it is C^1 and $df\colon U \times E \to F$ is C^k.

Remark

One can shows that f is C^k if and only if f is continuous, the iterated directional derivatives

$$d^j f(x, y_1, \ldots, y_j) := (D_{y_j} \cdots D_{y_1} f)(x)$$

exist at $x \in U$ for all $j \in \mathbb{N}$ with $j \leq k$ and $y_1, \ldots, y_j \in E$, and $d^j f\colon U \times E^j \to F$ is continuous.

Example. Every continuous linear map $\alpha\colon E \to F$ is C^k for all $k \in \mathbb{N}$ and hence smooth.

We already know that α is C^1 and that $d\alpha\colon E \times E \to F$ is the map

$$(x, y) \mapsto \alpha(y)\,.$$

Thus $d\alpha$ is a continuous linear map and hence C^k by induction. By definition, α is C^{k+1}.

Example. Every continuous bilinear map $\beta\colon E_1 \times E_2 \to F$ is C^k for all $k \in \mathbb{N}$ and hence smooth.

We already know that β is C^1 and that $d\beta\colon (E_1 \times E_2) \times (E_1 \times E_2) \to F$ is the map

$$((x_1, x_2), (y_1, y_2)) \mapsto \beta(y_1, x_2) + \beta(x_1, y_2)\,.$$

Thus $d\beta = \beta \circ \alpha_1 + \beta \circ \alpha_2$ with continuous linear maps $\alpha_1, \alpha_2\colon (E_1 \times E_2)^2 \to E_1 \times E_2$. The latter being C^k, also $d\beta$ is C^k, by the Chain Rule. Hence β is C^{k+1}.

Consider a locally convex algebra A (a locally convex space A, endowed with a continuous bilinear multiplication $A \times A \to A$ such that the assocative law $x(yz) = (xy)z$ holds and there exists a neutral element 1 for the multiplication, $1x = x1 = x$ for all $x \in A$.

Definition

If the unit group $A^\times := \{x \in A \colon (\exists y \in A)\; xy = yx = 1\}$ is open in A and the inversion map $\iota \colon A^\times \to A$, $x \mapsto x^{-1}$ is continuous, then A is called a **continuous inverse algebra** or **cia**.

Fact

If A is a **cia**, then the inversion map $\iota \colon A^\times \to A$ is smooth. Thus $A^\times$ is a Lie group.

Proof. For all $a, b \in A^\times$, we have

$$b^{-1} - a^{-1} = b^{-1}(a - b)a^{-1}\,.$$

Applying this to $a = x$ and $y = x + ty$ for $x \in A^\times$ and $y \in A$, we get

$$\frac{(x+ty)^{-1}-x^{-1}}{t} = (x+ty)^{-1}(-y)x^{-1} \to -x^{-1}yx^{-1}$$

as $t \to 0$. Thus $d\iota$ exists and

$$d\iota(x,y) = -x^{-1}yx^{-1} = -\iota(x)y\iota(x)$$

is continuous in (x,y), whence ι is C^1. If ι is C^k by induction, we deduce using the Chain Rule and smoothness of the bilinear algebra multiplication that $d\iota$ is C^k, whence ι is C^{k+1}. We also used that a map to a product is C^k if all of its components are so.

Example. Every Banach algebra A (e.g., $C^k([0,1],\mathbb{R})$) is a cia and hence $A^\times$ a Lie group.

Example. $C^\infty([0,1],\mathbb{R})$ is a cia.

§5 Comparison with calculus in normed spaces

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be Banach spaces, $U \subseteq E$ be an open subset and $f\colon U \to F$ be a map. Say that f is differentiable ast $x \in U$ if there exists a continuous linear map $f'(x)\colon E \to F$ such that the remainder term in the affine-linear approximation

$$f(y) = f(x) + f'(x)(y - x) + R(y)$$

satisfies
$$\lim_{y \to x} \frac{R(y)}{\|y - x\|_E} = 0\,.$$

The f is continuous at x and $f'(x)$ is unique as

$$f'(x)(y) = (D_y f)(x) \quad \text{for all } y \in E.$$

Call f **continuously Fréchet differentiable** (or FC^1) if f is differentiable at each $x \in U$ and $f'\colon U \to (\mathcal{L}(E,F), \|\cdot\|_{\mathrm{op}})$ is continuous. If f is FC^1 and f' is FC^k, say that f is FC^{k+1}.

Fact

If f is FC^k, then f is C^k. If f is C^{k+1}, then f is FC^k.

Notably, same notion of smoothness!

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