WGMP 2023: Summer School

Notable applications of OPS

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Białystok June 25-30, 2023

Recall: Let $\mathscr{M}$ be a space of Hermitean matrices ($M = \mathscr{M}^\dagger$) of size $n \times n$:
$\mathscr{M} := \{M \in Mat(n,n;\mathbb{C}),\ M_{ij} = M^\star_{ji}\}$

$$M_{ab} = X_{ab} + iY_{ab}\ ,\quad X_{ab} = X_{ba}\ ,\quad Y_{ab} = -Y_{ba} \tag{1}$$

$$\dim \mathscr{M} = \frac{n}{2}(n+1) + \frac{n}{2}(n-1) = n^2 \tag{2}$$

$$\mathrm{d}M := \prod_{a=1}^{n} \mathrm{d}X_{aa} \prod_{1\le a<b\le n} \mathrm{d}X_{ab}\,\mathrm{d}Y_{ab} \tag{3}$$

Probability measure:

$M = U \cdot \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} U^\dagger$

$$\mathrm{d}\mu(M) = \frac{1}{Z}\mathrm{e}^{-\Lambda \mathrm{Tr} V(M)}\,\mathrm{d}M = \frac{1}{Z}\mathrm{e}^{-\Lambda \sum_{a=1}^{n} V(x_a)}\,\mathrm{d}M = \Delta^2(\vec{x}) \prod \mathrm{d}x_i \cdot \mathrm{d}U \tag{4}$$

Z → normalization

Induced j.p.d.f. on eigenvalues:

Λ ↗ large parameter

$$\mu(\vec{x}) = \frac{1}{\mathscr{Z}} \prod_{1\le a<b\le n} (x_a - x_b)^2 \prod_{a=1}^{n} \mathrm{e}^{-\Lambda V(x_a)}\,\mathrm{d}x_a = \frac{1}{n!}\det\left[K_n(x_a,x_b)\right]_{1\le a,b\le n} \tag{5}$$

$\int_{\mathbb{R}} e^{-\Lambda V(x)} p_j(x) p_l(x)\,dx = \delta_{jl} h_l$

where

→ Christoffel Darboux Kernel

CD theorem

$$K_n(x,y) = \mathrm{e}^{-\Lambda\frac{V(x)+V(y)}{2}} \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{\|p_j\|^2} \overset{\downarrow}{=} \mathrm{e}^{-\Lambda\frac{V(x)+V(y)}{2}} \frac{p_n(x)p_{n-1}(y) - p_n(y)p_{n-1}(x)}{\|p_{n-1}\|^2 (x-y)} \tag{6}$$

Stat properties of Spect(M) as $n \to \infty$ $\Lambda \to \infty$ so that $\Lambda \sim tn$ $t > 0$

THEOREM 0.1

Suppose that a kernel $K(x,y)$ *has the properties of* **reproducibility** *and* **normalization** *(to* n*). Then*

n

$$\begin{vmatrix} K(x_1,x_1) & \cdots & K(x_1,x_r) \\ \vdots & & \\ K(x_r,1) & & K(x_r,x_r) \end{vmatrix}$$

$$\textbf{(a)} \qquad \int_{\mathbb{R}} \det[K(x_a,x_b)]_{a,b\le r}\, dx_r = (n-r+1)\det[K(x_a,x_b)]_{a,b\le r-1} \tag{7}$$

$$\textbf{(b)} \qquad \int_{\mathbb{R}^{n-r}} \det[K(x_a,x_b)]_{a,b\le n}\, dx_{r+1}\dots dx_n = (n-r)!\det[K(x_a,x_b)]_{a,b\le r-1} \tag{8}$$

REMARK 0.1

Dyson's theorem says that the JPDF and *all the marginals* (partial integrations thereof) are in the form of a **determinant** built out of the same kernel $\Rightarrow$ **determinantal random point fields** [17].

Remark 0.2

The whole statistical information is contained in the Kernel expressed by orthogonal polynomials.

Example 0.1 (Density of eigenvalues)

From the JPDF we integrate all variables except one; this gives the **density** of eigenvalues (i.e. the expected number of eigenvalues in the interval $[x, x+dx]$. According to Dyson's theorem

$$\rho_n(x) = K_n(x,x) = \mathrm{e}^{-\lambda V(x)} \sum_{j=0}^{n} \frac{p_j(x)^2}{\|p_j\|^2} \tag{9}$$

Example 0.2 (Pair correlations of eigenvalues)

From the JPDF we integrate all variables except 2; this gives the **correlations** of eigenvalues (i.e. the expected number of eigenvalues in the interval $[x, x+dx] \times [y, y+dy]$.

$$\rho_n^{(2)}(x\,y) = \det \begin{bmatrix} K_n(x,x) & K_n(x,y) \\ K_n(y,x) & K_n(y,y) \end{bmatrix} = K_n(x,x)K_n(y,y) - K_n(x,y)^2 \tag{10}$$

Example: Gaussian Unitary Ensemble

Consider the simplest example $V = \frac{x^2}{2}$ $\Lambda = N = n$

$$\mathrm{d}\mu(M) = \frac{1}{Z_n} \mathrm{e}^{-\frac{N}{2}\mathrm{Tr}M^2}\,\mathrm{d}M \xrightarrow{\text{reduced meas.}} \mathrm{d}\mu_r(\vec{x}) = \prod_{j<k}(x_j - x_k)^2 \prod_{j=1}^{n} \mathrm{e}^{-\frac{N}{2}x_j^2}\,\mathrm{d}x_j \tag{11}$$

Note: the entries of M_{ij} are **independent and normal** but the eigenvalues x_j are **not independent**.
If $H_n(x)$ are the **(monic) Hermite orthogonal polynomials**

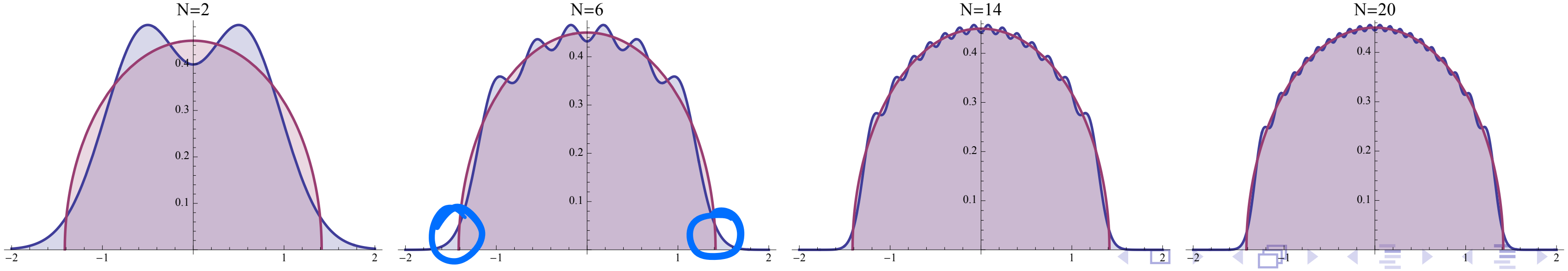
$$\int_{\mathbb{R}} H_n(x)H_m(x)\mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x = h_n\delta_{nm}\ , \quad H_n(x) = x^n + \dots\ , \quad h_n = \sqrt{2\pi}n! \tag{12}$$

then the kernel is

$$K_n(x,y) = \mathrm{e}^{-\frac{N}{4}(x^2+y^2)} \sum_{j=0}^{n-1} \frac{\sqrt{N}}{\sqrt{2\pi}n!} H_n(\sqrt{N}x)H_n(\sqrt{N}y) \tag{13}$$

The **density** of eigennvalues can be computed in closed form and has a limit as $n = N \to \infty$ given by the **Wigner semicircle law**

$$\rho_W(x) = \frac{1}{\pi}\sqrt{2 - x^2} \tag{14}$$



Consider now the **positive semidefinite** Hermitean matrices M with the probability measure

$$\mathrm{d}\mu(M) = \frac{1}{Z_n}(\det M)^\alpha \mathrm{e}^{-N\mathrm{Tr}M}\,\mathrm{d}M \xrightarrow{\text{reduced meas.}} \mathrm{d}\mu_r(\vec{x}) = \prod_{j<k}(x_j - x_k)^2 \prod_{j=1}^{n} \boxed{x^\alpha \mathrm{e}^{-Nx_j}}\,\mathrm{d}x_j \tag{15}$$

If $L_n(x)$ are the Laguerre orthogonal polynomials:

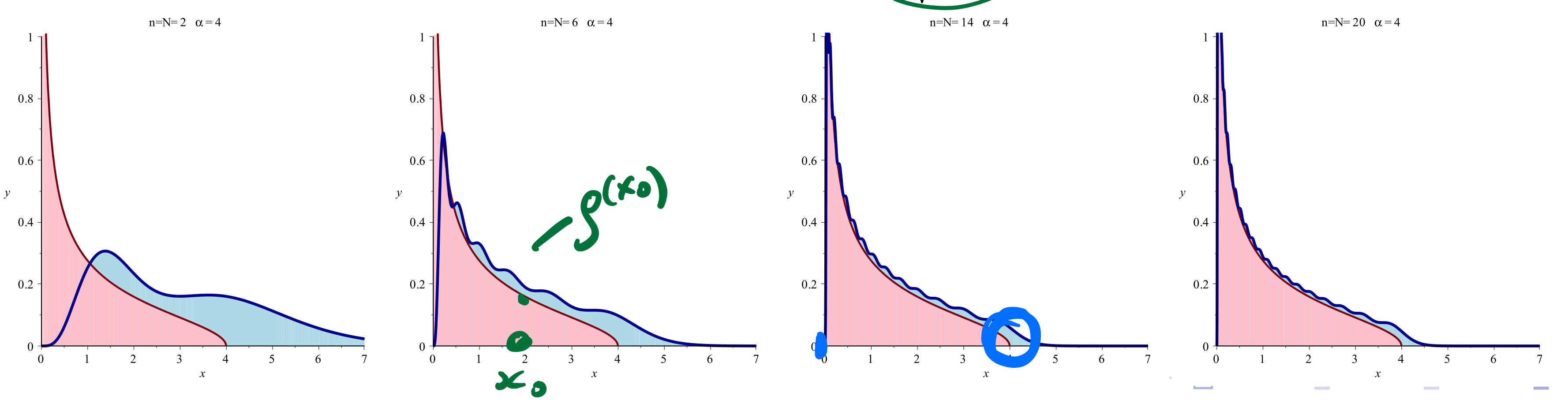
$$\int_{\mathbb{R}_+} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) x^\alpha \mathrm{e}^{-x}\,\mathrm{d}x = h_n \delta_{nm}\ , \qquad h_n = \frac{\Gamma(n+\alpha+1)}{n!} \tag{16}$$

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}e^x}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right) = x^{-\alpha}\,\frac{\left(\frac{d}{dx}-1\right)^n}{n!}\,x^{n+\alpha} \tag{17}$$

then the kernel is

$$K_n^{(\alpha)}(x,y) \ :=N^{\alpha+1}\mathrm{e}^{-\frac{x+y}{2}}\Gamma(\alpha+1)\sum_{i=0}^{n}\frac{L_i^{(\alpha)}(Nx)L_i^{(\alpha)}(Ny)}{\binom{\alpha+i}{i}} \tag{18}$$

and the density tends asymptotically to $\rho_\infty(x) = \frac{1}{\pi}\sqrt{\frac{4-x}{x}}$ (**Marchenko-Pastur**):



For the GUE (Hermite case) one can prove the
Sine-kernel (Universality) in the Bulk

expected dist of evals

$$\lim_{n\to\infty} \frac{1}{n\rho(x_0)} K\left(x_0 + \frac{\xi}{n\rho(x_0)}, x_0 + \frac{\eta}{n\rho(x_0)}\right) = \frac{\sin(\pi(\eta-\xi))}{\pi(\eta-\xi)} \tag{19}$$

Airy-kernel (Universality) at the Edge: (for Hermite)

$$\lim_{N\to\infty} N^{\frac{1}{3}} \frac{\sqrt{2}}{2} K_N\left(\sqrt{2} + \frac{\sqrt{2}\xi}{2N^{\frac{2}{3}}}, \sqrt{2} + \frac{\sqrt{2}\eta}{2N^{\frac{2}{3}}}\right) = \frac{\mathrm{Ai}(\xi)\mathrm{Ai}'(\eta) - \mathrm{Ai}'(\xi)\mathrm{Ai}(\eta)}{\xi - \eta} \tag{20}$$

$$= K_{\mathrm{Ai}}(\xi,\eta) \tag{21}$$

The notion of universality is akin to the *Central limit theorem* in statistics:

$$\frac{\sum_1^N X_j - N\overline{X_j}}{\sigma\sqrt{N}} \to N(0,1) \tag{22}$$

where X_j are IID random variables (with finite second moment $<(X_j - \overline{X_j})^2 >= \sigma^2$)
Note the scaling and the scale of the fluctuations (i.e. $\sqrt{N}$).
For RMM, the universality classes exhibit themselves in suitable scaling *regimes* and spectral regions.

Universality (edge)

We are "zooming in" to the **edge of the spectrum** at $x = \sqrt{2}$; for example the density of eigenvalues is

$$N^{\frac{1}{3}} \frac{\sqrt{2}}{2} \rho_N \left(\sqrt{2} + \frac{\sqrt{2}}{2\mathbf{N}^{\frac{2}{3}}} \xi \right) \xrightarrow[N\to\infty]{} \left(Ai'(\xi)\right)^2 - \xi Ai^2(\xi) \tag{23}$$

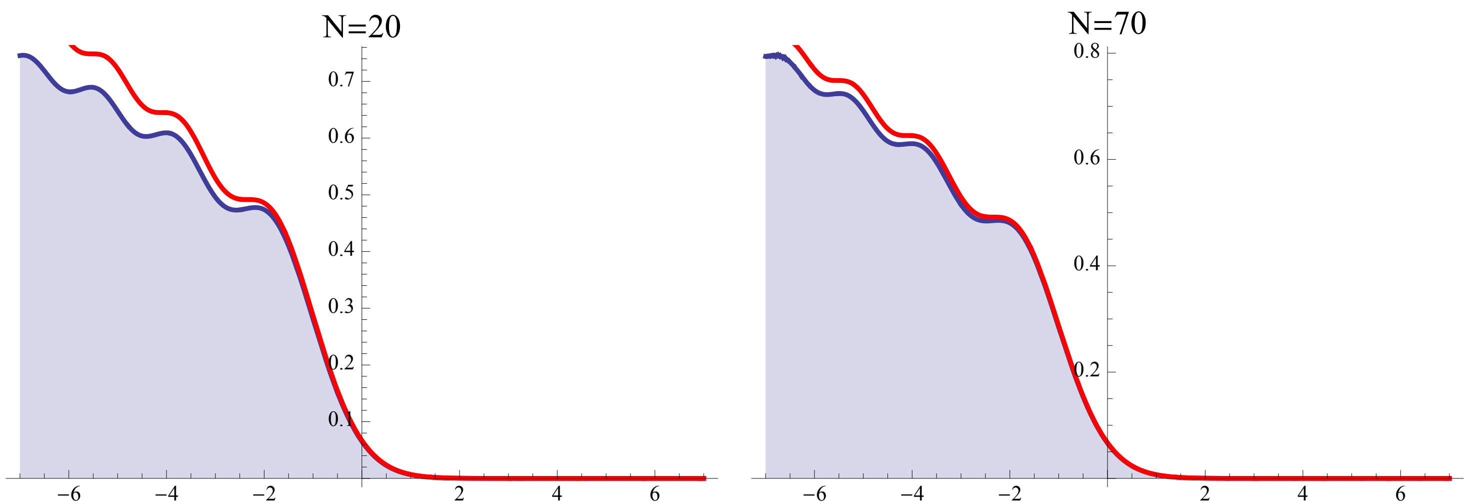


FIGURE: Comparison between the actual density and the Airy density (in red)

The first proof that the kernels of any (generic) Random Matrix Unitary model enjoys the same Sine kernel and Airy kernel universal behaviour (independently of the details of the potential V) was the main accomplishment of the theory of OPS and the nonlinear steepest descent method in the late '90 by Deift-Kriecherbauer-McLaughlin-Venakides-Zhou [12].

Application: Toda equations

Dynamical system n particles $x_1, \ldots, x_n$ on the line

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{k=1}^{n-1} \mathrm{e}^{x_k - x_{k-1}} \tag{24}$$

$$\dot{x}_k = p_k, \qquad \dot{p}_k = \ddot{x}_k = \mathrm{e}^{x_{k-1}-x_k} - \mathrm{e}^{x_k - x_{k+1}} \tag{25}$$

Flaschka-Manakov: introduce new variables

$$a_k := -\frac{p_k}{2}, \qquad b_k = \frac{1}{2}\mathrm{e}^{\frac{x_k - x_{k-1}}{2}} \tag{26}$$

then arrange them into matrices:

$$J := \begin{bmatrix} a_1 & b_1 & 0 & \ldots & 0 \\ b_1 & a_2 & b_2 & \ddots & \vdots \\ 0 & b_2 & a_3 & \ddots & 0 \\ \vdots & \ddots & & \ddots & b_{n-1} \\ 0 & \ldots & 0 & b_{n-1} & a_n \end{bmatrix}, B := \begin{bmatrix} 0 & b_1 & 0 & \ldots & 0 \\ -b_1 & 0 & b_2 & \ddots & \vdots \\ 0 & -b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & & \ddots & b_{n-1} \\ 0 & \ldots & 0 & -b_{n-1} & 0 \end{bmatrix} \tag{27}$$

The equations of motion become:

$$\frac{d}{dt}\det\left[L(t)\right] = \operatorname{tr}\left(L^{adj}(t) \cdot \frac{dL}{dt}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}J = [B,J] \quad \Rightarrow \quad \text{The spectrum of } J \text{ is conserved} \tag{28}$$

which are equivalent to

$$\left(\text{If } \dot{M}(t) = [M,K] \Rightarrow \frac{d}{dt}\left(\det(\lambda - M(t))\right) \equiv 0\right.$$

$$\dot{b}_j = b_j\left(a_{n+1} - a_n\right) \quad \dot{a}_n = 2\left(b_n^2 - b_{n-1}^2\right). \tag{29}$$

Suppose now we have infinitely many particles and arrange them into the same (semi-infinite) matrices: multiply J on the left by D and on the right by D^{-1} with

$$D := \operatorname{diag}(1, b_1, b_2, \dots, b_n, \dots) \tag{30}$$

$$\widetilde{J} = \begin{bmatrix} a_1 & 1 & 0 & \dots & 0 \\ b_1^2 & a_2 & 1 & \ddots & \vdots \\ 0 & b_2^2 & a_3 & \ddots & 0 \\ \vdots & \ddots & & \ddots & 1 \\ 0 & \dots & 0 & b_{n-1}^2 & a_n \end{bmatrix} \tag{31}$$

This is the Jacobi matrix describing a three-term recurrence relation! By Favard's theorem, there is a measure $\mathrm{d}\mu(z;t)$ (depending on t!) whose monic OPS satisfies

$$z p_n(z) = p_{n+1}(z) + a_{n+1} p_n(z) + b_n^2 p_{n-1}(z), \quad n = 0, 1, \ldots . \tag{32}$$

$p_{-1} = 0$
$p_0 = 1$

$$\int_{\mathbb{R}} p_n(z) p_m(z)\, \mathrm{d}\mu(z;t) = h_n(t)\delta_{nm} \tag{33}$$

THEOREM 0.2

The measure $\mathrm{d}\mu(z;t)$ evolves as follows:

$$\mathrm{d}\mu(z;t) = \mathrm{e}^{2zt}\, \mathrm{d}\mu(z;0) \tag{34}$$

Thus, the solutions of the semi–infinite Toda lattice chain are parametrized by positive measures on the real line.

REMARK 0.3 (HINT ON HOW TO PROVE IT)

The ortho**normal** polynomials satisy the relation (in matrix form) $z\vec{\pi}(z;t) = J(t)\vec{\pi}(z;t)$.

- Show that $\partial_t \vec{\pi} = -(J_0 + 2J_-)\vec{\pi}$, where $J = J_+ + J_0 + J_-$ are the three diagonals.
- Compute $\partial_t J$ and compare with $[J, B]$ where $B = J_+ - J_-$.

$$\vec{\pi} = \begin{bmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_n \\ \vdots \end{bmatrix} \qquad \int \pi_\ell \pi_m \, d\mu = \delta_{\ell m}$$

OPS AND KP

In the Toda example the equations can be shown to be equivalent to the following for the Hankel matrix of moments (note change of letter $x = 2t$ for convenience)

$$\tau_n(x) := \det\left[\int_{\mathbb{R}} z^{a+b} e^{xz}\, d\mu(z)\right]_{a,b=0}^{n-1}; \tag{35}$$

$$\partial_t^2 \ln \tau_n(x) = \frac{\tau_{n+1}(x)\tau_{n-1}(x)}{\tau_n(x)^2} \tag{36}$$

(Hirota bilinear form)
Consider an arbitrary measure (e.g. cpctly supported) $d\mu(z)$ and define

$$\mu_j(x,y,t) := \int_{\mathbb{R}} z^j e^{xz+yz^2+tz^3}\, d\mu(z) \tag{37}$$

Define similarly

$$\tau_n(x,y,t) := \det\left[\int_{\mathbb{R}} z^{a+b} e^{xz+y\frac{z^2}{\cancel{2}}+t\frac{z^3}{\cancel{3}}}\, d\mu(z)\right]_{a,b=0}^{n-1} = \det\left[\mu_{a+b}(x,y,t)\right]_{a,b=0}^{n-1} \tag{38}$$

Then one can prove

THEOREM 0.3

The function $u(x,y,t) := 2\partial_x^2 \ln \tau_n(x,y,t)$ satifies the Kadomtsev-Petviashvili equation

$$\boxed{\left(4u_t - 6uu_x - u_{xxx}\right)_x = 3u_{yy}} \tag{39}$$

The Painlevé equations are a class of 2nd order ODEs

$$y'' = R(y', y, t) \in \mathbb{C}[y'] \otimes \frac{\mathbb{C}[y,t]}{\mathbb{C}[y,t]} \tag{40}$$

that appear recurrently in math-phys: they have the

PAINLEVÉ PROPERTY (SCHEMATIZING)

The member of the general family of solutions has only poles for its *movable singularities* (i.e., singularities whose position is not already evident in the equation and depend on the initial conditions).

1. $$\frac{d^2y}{dt^2} = 6y^2 + t$$
2. $$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$
3. $$ty\frac{d^2y}{dt^2} = t\left(\frac{dy}{dt}\right)^2 - y\frac{dy}{dt} + \delta t + \beta y + \alpha\frac{y^3}{t} + \gamma\frac{y^4}{t}$$
4. $$y\frac{d^2y}{dt^2} = \tfrac{1}{2}\left(\frac{dy}{dt}\right)^2 + \beta + 2(t^2 - \alpha)y^2 + 4ty^3 + \tfrac{3}{2}y^4$$
5. $$\frac{d^2y}{dt^2} = \left(\frac{1}{2y} + \frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{t} + \delta\frac{y(y+1)}{y-1}$$
6. $$\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{y-t}\right)\frac{dy}{dt} + \frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{y^2} + \gamma\frac{t-1}{(y-1)^2} + \delta\frac{t(t-1)}{(y-t)^2}\right)$$

Rational solutions

Except PI, all have special **rational** (but also algebraic and other types) solutions.

$$\frac{\mathrm{d}^2 u(t)}{\mathrm{d}t^2} = 2u(t)^3 + tu(t) + \alpha, \tag{41}$$

Rational solution iff $\alpha = n \in \mathbb{Z}$;

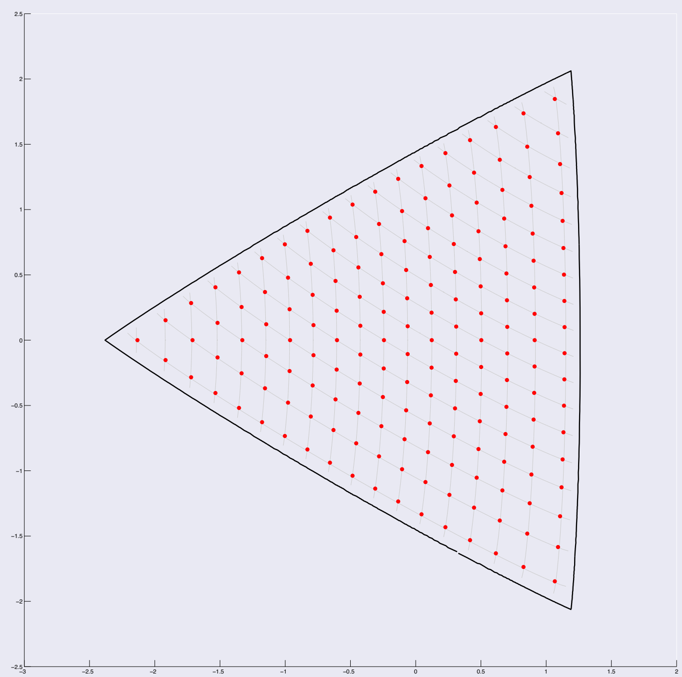
$$u_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{Y_{n-1}(t)}{Y_n(t)} \tag{42}$$

with Y_n the Vorob'ev–Yablonski polynomials of degree $n(n+1)/2$.

$$Y_{n+1}(t)Y_{n-1}(t) = tY_n^2(t) - 4\left[Y_n''(t)Y_n(t) - \left(Y_n'(t)\right)^2\right], \qquad n \geq 1,\ t \in \mathbb{C} \tag{VY}$$

with $Y_0(t) = 1, Y_1(t) = t$.

The regularity of the pattern of zeroes of $Y_n(t)$ observed numerically by Clarkson ['03], and explained (asymptotically and analytically) by Buckingham–Miller ['14], B–Bothner ['14]; generalized to the hierarchy in Balogh-B–Bothner ['15].



THEOREM 0.4 ([5])

For any $n \geq 1$, we have

$$Y_{n-1}^{2}(x) = (-)^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2^{n-1}} \prod_{k=1}^{n-1} \left[\frac{(2k)!}{k!} \right]^{2} \det \left[\mu_{\ell+j-2}(x) \right]_{\ell,j=1}^{n}. \tag{43}$$

where $\lfloor y \rfloor$ denotes the floor function of a real number y and

$$\exp\left[-\frac{t^{3}}{3} + tx \right] = \sum_{k=0}^{\infty} \mu_{k}(x) t^{k} \tag{44}$$

or equivalently

$$\mu_{j}(x) = \oint_{|\zeta|=1} \zeta^{j} \, \mathrm{e}^{\frac{1}{3\zeta^{3}} - \frac{x}{\zeta}} \frac{\mathrm{d}\zeta}{2i\pi\zeta} \tag{45}$$

Where are the poles?

- Poles of rational solutions are zeros of V-Y polynomials;
- Squares of V-Y are Hankel determinants of moments;
- Hankel dets' of moments are zero iff the n-th OP does not exist.

Thus:

- Study the solvability (for large n) of the Fokas-Its-Kitaev RHP; this can be done via the Deift–Zhou method;
- Poles can only occur if the number of cuts of the g-function ≥ 2

$$p_n(z) = \frac{\det\begin{bmatrix} \cdots & & & \cdots \\ 1 & & & z^n \end{bmatrix}}{\text{Hankel}}$$

Starz!

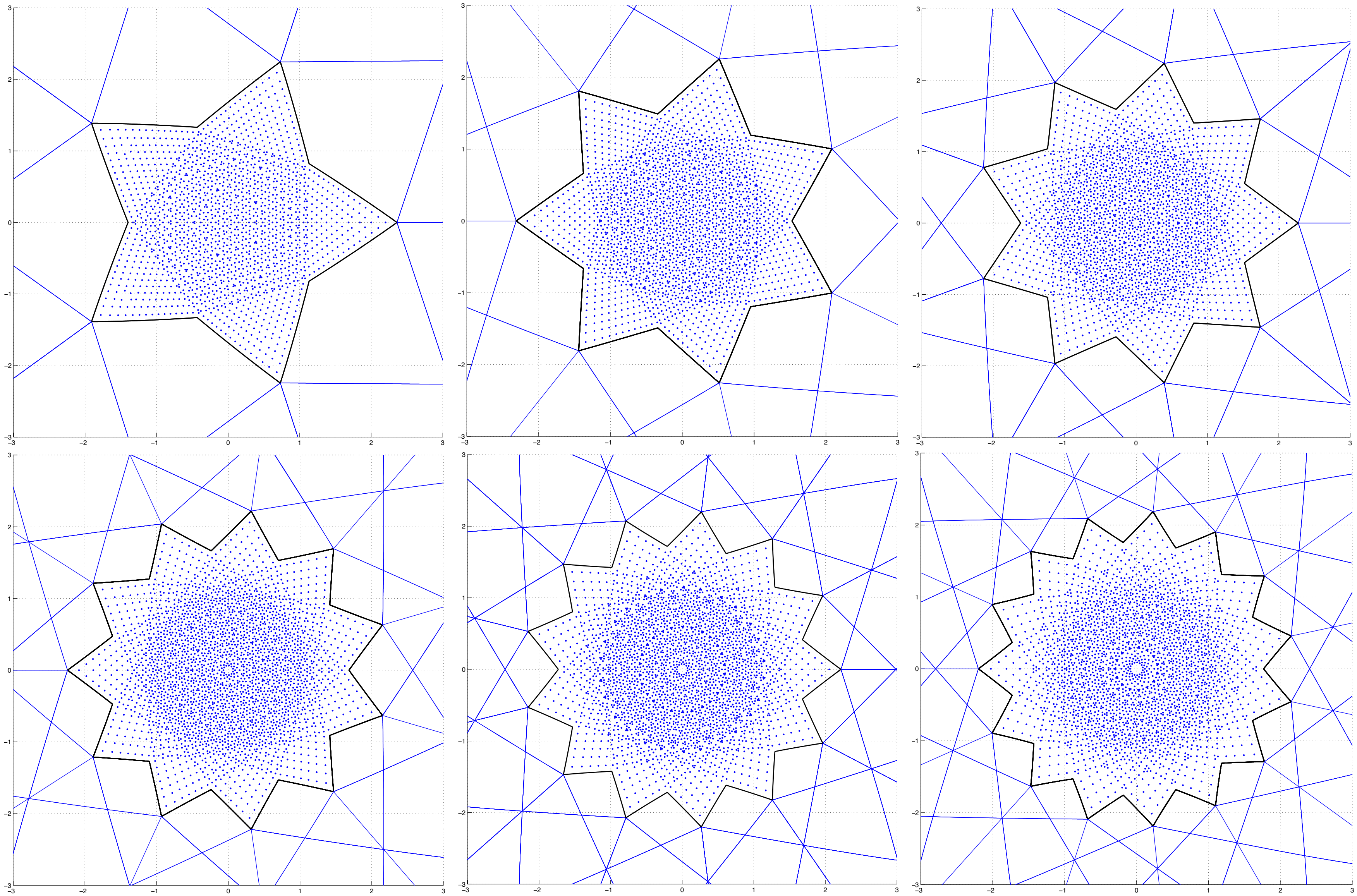


Figure: The roots of the rescaled higher Vorob'ev-Yablonski polynomials, corresponding to rational solutions of higher versions of the PII equation.

There are generalizations

1. Multiple orthogonal polynomials: related to multi-matrix models, simultaneous Padé approximations,
2. OPS emerge from Padé problems (approximation by rational functions). If the function we approximate is periodic it is natural to use periodic functions (ratio of periodic functions). If the function is elliptic (doubly periodic), it is natural to use ratio of elliptic functions $\wp, \wp'$ etc.
   A new theory of Padé approximation theory (orthogonal sections) on higher genus surfaces begins [2, 3, 4]!
3. The topic is connected with **potential theory** on the plane and on algebraic surfaces [8]

$$W(z) = \frac{Q_{n-1}}{P_n} + O\left(z^{-2n-1}\right)$$

$P_n \to$ OPS

$$W(z) = \int \frac{d\mu(x)}{z-x} \qquad \text{(cpctly supported)}$$

$$= \frac{\mu_0}{z} + \frac{\mu_1}{z^2} + \dots$$

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