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An introduction the Deift-Zhou steepest descent approach to asymptotics of OPS

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THE HOW

For all classical OPS (Hermite, Laguerre, Jacobi, Charlier, et al.) the n–th polynomial admits an **integral representation**. For example Hermite:

$$H_n(x) = \frac{n!}{2^n} \oint_{|w|=1} w^{-n-1} e^{2xw-w^2} \frac{dw}{2i\pi} \tag{1}$$

Upon rescaling

$$h_n(x) := n^{-\frac{n}{2}} H_n(\sqrt{n}x) \tag{2}$$

$$h_n(x) = \frac{n!n^{-\frac{n}{2}}}{2^n 2i\pi} \oint_{|w|=1} w^{-n-1} e^{2\sqrt{n}xw-w^2} dw = \frac{n!n^{-n}}{2^n 2i\pi} \oint_{|\xi|=1} \xi^{-n-1} e^{-n(\xi^2-2x\xi)} d\xi = \tag{3}$$

$$= \frac{n!n^{-n}}{2^n 2i\pi} \oint_{|\xi|=1} e^{-n(\xi^2-2x\xi+\ln\xi)} \frac{d\xi}{\xi} \tag{4}$$

Then the asymptotics proceeds from a version of the **steepest descent analysis** of integrals with a parameter (classical).

1. Write the integral in the form

$$\int_{\Gamma} g(z)\mathrm{e}^{-\Lambda V(z)}\,\mathrm{d}z \tag{5}$$

   where $\Lambda \to +\infty$ is the **large parameter** and Γ is a contour in the complex plane homotopic to the original one in $\mathbb{C}$ minus all singularities of $g, \mathrm{e}^{\Lambda V(z)}$.

2. The integral is dominated (a version of **stationary phase** arguments) by pieces of Γ near *some of the* **critical points**. Formally, if x_0 is a critical point $V'(x_0) = 0$ then (assuming $V''(x_0) \neq 0$)

$$\int_{\Gamma} g(z)\mathrm{e}^{-\Lambda V(z)}\,\mathrm{d}z \sim g(x_0)\mathrm{e}^{-\Lambda V(x_0)}\sqrt{\frac{2\pi}{\Lambda V''(x_0)}}\left(1+\mathscr{O}(\Lambda^{-1})\right) \tag{6}$$

   The dominant behaviour is the one given by the critical point where $\Re V(x_0)$ is maximal (if several have the same, we take the sum of as many terms).

3. The real problem is to decide which critical points are really to be used. This depends on the following:

Find a homotopy representative of Γ that crosses only those critical points in a direction of steepest descent, where $\Re V$ has a local/global min.

Landscape for Hermite's case: $V(\xi) = \xi^2 - 2x\xi + \ln \xi$

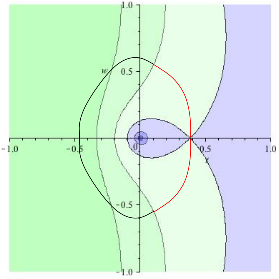


FIGURE: The red arcs are where the imaginary part is constant and the real part has a minimum at the critical point. The black arcs have the property that the real part of $V(\xi)$ is strictly bigger. If we put the "sea level" of the real part at zero where the critical point $\xi = \frac{x - \sqrt{x^2-2}}{2}$ is, the landscape is the one depicted. The blue parts are the sea (at different depths), the green parts are the "land".

- For general OPS we do not have an integral representation and classical steepest descent analysis is not a viable strategy.
- Starting with the seminal works of Fokas-Its-Kitaev and the fundamental asymptotic methods of Deift-Zhou in the late 90's a new paradigm for studying OPS (and many other problems based on the inverse-scattering method) has emerged based on the notion of **Riemann Hilbert Problem**.
- The method is nowadays referred to variably as the "Riemann-Hilbert method" the "nonlinear steepest descent method" or the "Deift-Zhou method".
- Riemann Hilbert problems (RHPs for short) are a class of boundary-value problems in the complex plane for matrix–valued functions.
- The topic has distant ramifications and extends also to the theory of vector bundles on Riemann surfaces (which are a particular instance of RHPs).
- In general there is a cultural dichotomy in the community, by which analysis oriented researchers speak of RHPs and algebro-geometric researchers of "clutching functions", vector bundles etc.
- The focus is also different in the respective contexts. Here we present the minimum to illustrate the ideas in the context of OPs.

Let $\Sigma = \mathbb{R}$ and $\mathrm{d}\mu(x) = \mathrm{e}^{-\Lambda V(x)}\,\mathrm{d}x$ with $V(x)$ (the **potential**) bounded from below and growing sufficiently fast at ∞. For example $V(x) = x^2$ (Hermite).

Problem 0.1 (The RHP for Orthogonal Polynomials: Fokas-Its-Kitaev '90)

Find a 2×2 matrix–valued function $Y(z) = Y_n(x)$ with the properties

1. $Y(z)$ is analytic in $\mathbb{C}_\pm := \{\pm\Re(z) > 0\}$;
2. The boundary values of $Y(z)$ on the real axis (oriented in the natural direction) satisfy

$$Y_+(x) = Y_-(x) \overbrace{\begin{bmatrix} 1 & \frac{\mathrm{d}\mu(x)}{\mathrm{d}x} \\ 0 & 1 \end{bmatrix}}^{=:M(x)} \tag{7}$$

3. In the sectors $\arg(z) \in (0,\pi)$ and $\arg(z) \in (\pi, 2\pi)$ the function $Y(z)$ has the expansion

$$Y(z) = \left(\mathbf{1} + \mathscr{O}\left(\frac{1}{z}\right)\right) \begin{bmatrix} z^n & 0 \\ 0 & z^{-n} \end{bmatrix} = (\mathbf{1} + \mathscr{O}(z^{-1}))z^{n\sigma_3} \ , \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{8}$$

   The above expansion is *uniform*[a] in the sense that for any $R > 0$ there exists $C > 0$ such that for $|z| > R$, $z \notin \mathbb{R}$,

$$\left\| Y(z) z^{-n\sigma_3} - \mathbf{1} \right\| < C \frac{1}{|z|} \tag{9}$$

[a] This is not the strongest form of the problem but it is sufficient for us.

PROPOSITION 0.1 (UNIQUENESS)

If a solution of Prob. 0.1 exists, then it is unique.

Proof. ..... ∎

THEOREM 0.1

Problem (0.1) admits a unique solution of the form

$$Y_n(z) := \begin{bmatrix} p_n(z) & \frac{1}{2i\pi}\int_{\mathbb{R}} \frac{p_n(x)\mathrm{e}^{-\Lambda V(x)}\,\mathrm{d}x}{x-z} \\ \frac{-2i\pi}{h_{n-1}} p_{n-1}(z) & \frac{-1}{h_{n-1}}\int_{\mathbb{R}} \frac{p_{n-1}(x)\mathrm{e}^{-\Lambda V(x)}\,\mathrm{d}x}{x-z} \end{bmatrix} \tag{10}$$

where p_n, p_{n-1} *are the* monic *orthogonal polynomials for the measure* $\mathrm{e}^{-\Lambda V(x)}\,\mathrm{d}x$

The proof requires the use of the Sokhotski-Plemelji formula.
The point is that the $(1,1)$ entry contains the n-th OP. The typical asymptotic analysis consists of

1. find strong asymptotic (pointwise) as $n \to \infty$ in all regions of the complex plane z;
2. same as above but with concurrent limit $\Lambda \to \infty$ and $\Lambda \simeq tn$ $(t > 0)$. This is known as "scaling behaviour asymptotics" or "with varying weight".

Some key moments of the DZ method

To get an idea of the method, it consists in

1. "massage" the problem into an equivalent one, a sort of "deformation of contours" The first step is to change the unknown matrix $Y_n(x)$ by an explicit transformation
$$W_n(z) := \mathrm{e}^{-\frac{n}{2}\ell\sigma_3} Y(z) \mathrm{e}^{-n\left(g(z)-\frac{\ell}{2}\right)\sigma_3} \tag{11}$$
where ℓ is a constant and $g(z)$ an appropriate *scalar* function obtained from a functional variational problem (see below).
2. After some more steps, the matrix W can be schematically written in the form
$$W_n(z) = \mathscr{E}(z;n)P(z;n) \tag{12}$$
where $P(z)$ is an **explicit** matrix function and $\mathscr{E}$ is a "remainder" term, namely a matrix which can be proved to converge **pointwise** to the identity matrix as $n \to \infty$.

Given $V(x)$ as above

Theorem 0.2 (e.g. in Saff–Totik's book, Ch. 1 [10])

There is a unique **probability measure** $\rho(x)\,\mathrm{d}x$ *minimizing*

$$\mathscr{F}[\mathrm{d}\mu] := \int_{\mathbb{R}} V(x)\,\mathrm{d}\mu(x) + \int_{\mathbb{R}^2} \ln\frac{1}{|x-y|}\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \tag{13}$$

The minimizer $\rho(x)\,\mathrm{d}x$ *is characterized by*

$$V(x) + 2\int_{\mathbb{R}} \ln\frac{1}{|x-y|}\rho(y)\,\mathrm{d}y + \ell \geq 0 \qquad x\in\mathbb{R} \tag{14}$$

$$V(x) + 2\int_{\mathbb{R}} \ln\frac{1}{|x-y|}\rho(y)\,\mathrm{d}y + \ell \equiv 0 \qquad x\in\operatorname{supp}\rho \tag{15}$$

The constant ℓ *is called* **(modified) Robin's constant***.*

Electrostatic interpretation

The functional $\mathscr{F}$ describes the energy of a distribution of "electrons" on a wire ($\mathbb{R}$) subject to

- Mutual repulsion with the Coulomb potential of the plane $\ln\frac{1}{|x-x'|}$;
- Confining external potential $V(x)$
- in the limit we expect a continuous distribution $\mathrm{d}\mu(x)$ and the most probable one is the minimizer of the total electrostatic energy.

THEOREM 0.3 (DEIFT ET AL.)

Suppose $V(x)$ is also real-analytic: then $\operatorname{supp}\rho$ is a **finite union of compact intervals**.

A simple proof is available (using Shiffer's variations). It can also be shown that if $V(x)$ is **convex** (concave upwards) then there is only one interval of support. Since additional technical complications arise when there are several intervals, we shall assume that the support is indeed only one single interval

$$V''(x) > 0 \quad \Rightarrow \quad \operatorname{supp}\rho = [a,b] \tag{16}$$

DEFINITION 0.1 (THE g–FUNCTION)

$$g(z) := \int_a^b \ln(z-y)\rho(y)\,\mathrm{d}y \tag{17}$$

where $g(z)$ is defined as analytic on $\mathbb{C}$ minus the cut from $-\infty$ to b, with the principal branch of ln; for z approaching $\mathbb{R}$ above/below:

$$\ln(z_\pm - y) = \ln|z-y| \pm i\pi\chi_{y\geq z} \tag{18}$$

So that for $z = x \in \mathbb{R}$

$$g_\pm(x) = \int_a^b \ln|x-y|\rho(y)\,\mathrm{d}y \pm i\pi\chi_{x\leq b}\int_x^b \rho(y)\,\mathrm{d}y \tag{19}$$

Assuming that we know existence of the equilibrium measure (and sufficient smoothness) we want to find the solution of the **scalar** RHP

$$V(x) - g_+(x) - g_-(x) + \ell = 0 \quad \Rightarrow \quad g'_+(x) + g_-(x) = V'(x)\ , \quad x \in [a,b] \tag{20}$$

The following analysis is *perfunctory*:

let $R(z) := \sqrt{(z-a)(z-b)}$ be the holomorphic function on $\mathbb{C} \setminus [a,b]$ with $R(z) \sim z$ at infinity. Then (from the argument principle)

$$R_+(x) = -R_-(x). \tag{21}$$

Dividing (20) by R_+ we have

$$\frac{1}{R_+}(g'_+ + g'_-) = \left(\frac{g'}{R}\right)_+ - \left(\frac{g'}{R}\right)_- = \frac{V'}{R_+} \tag{22}$$

Thus the function $f := g'/R$ is analytic on $\mathbb{C} \setminus [a,b]$ and

$$f_+(x) - f_-(x) = \frac{V'(x)}{R_+(x)} \qquad x \in [a,b] \tag{23}$$

This RHP is solved with the **Sokhotsky-Plemelji** formula

$$f(z) = \int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} \quad \Rightarrow \quad g'(z) = R(z)\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} \tag{24}$$

On the other hand we had $g(z) = \ln(z) + \mathcal{O}(z^{-1})$ and hence $g'(z) = \frac{1}{z} + \mathcal{O}(z^{-2})$. The expansion of the proposed expression at $z = \infty$ is

$$g'(z) = R(z) \int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} = -\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)2i\pi} + \frac{1}{z}\left(\frac{b+a}{2}\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)2i\pi} - \int_a^b \frac{xV'(x)\,\mathrm{d}x}{R_+(x)2i\pi}\right) + \dots$$

This gives the following two equations (*moment conditions*) for the two unknowns a, b

$$-\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)2i\pi} = 0 \qquad -\int_a^b \frac{xV'(x)\,\mathrm{d}x}{R_+(x)2i\pi} = 1 \tag{26}$$

For V polynomial, both integrals are computed explicitly and the equations become algebraic.

Example 0.1

For $V(x) = \frac{t}{2}x^2 + \frac{\kappa}{4}x^4$ one obtains (**exercise!**) $b = -a$ and

$$a = \left(\frac{-2t + \sqrt{4t^2 + 48\kappa}}{3\kappa}\right)^{\frac{1}{2}} \tag{27}$$

Another form is as follows: if γ is a counterclockwise contour surrounding $[a,b]$ then the residue theorem yields

$$g'(z) = R(z)\int_a^b \frac{V'(x)\,\mathrm{d}x}{R_+(x)(x-z)2i\pi} = -\frac{1}{2}R(z)\oint_\gamma \frac{V'(x)\,\mathrm{d}x}{R(x)(x-z)2i\pi} = \tag{28}$$

$$= \frac{V'(z)}{2} - \frac{1}{2}R(z)\oint_{|x|>|z|} \frac{V'(x)\,\mathrm{d}x}{R(x)(x-z)2i\pi} = \tag{29}$$

$$= \frac{V'(z)}{2} - \frac{1}{2}R(z)\oint_{|x|>|z|} \frac{(V'(x)-V'(z))\,\mathrm{d}x}{R(x)(x-z)2i\pi} = \frac{V'(z)}{2} - M(z)R(z) \tag{30}$$

where $M(z)$ is patently a polynomial of degree at most $\deg V - 2$.

Since the equilibrium density is $\rho(x) = i\frac{g'_+(x)}{\pi}$ we see that

$$\rho(x) = \frac{1}{\pi}M(x)\sqrt{|x-a|\,|x-b|} \tag{31}$$

and hence $M(z)$ must remain positive for $x \in [a,b]$.

EXAMPLE 0.2

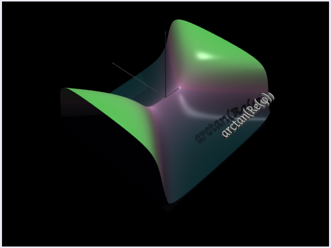
For $V = \frac{x^2}{2}$ the OP's involved are the Hermite polynomials: the equilibrium density is

$$\rho(x) = \frac{1}{\pi}\sqrt{2 - x^2}\ , \quad x \in [-2, 2] \tag{32}$$

and the complex effective potential φ

$$\varphi = \frac{z\sqrt{z^2 - 2}}{2} - \ln\left(\frac{z + \sqrt{z^2 - 2}}{2}\right) \tag{33}$$

The plot of $\arctan(\Re\varphi)$ is below: note that $\Re\varphi = \equiv 0$ on the support $\operatorname{supp}\rho = [-2, 2]$.



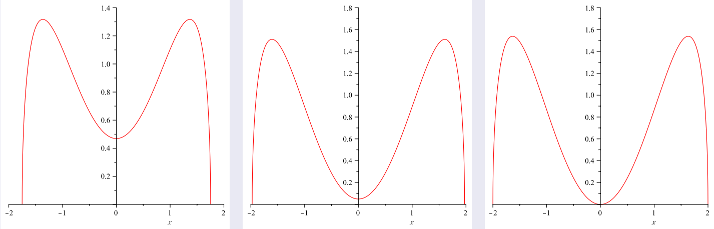
Example 0.3

In the above example $V(x) = \frac{t}{2}x^2 + \frac{\kappa}{4}x^4$ one finds (**exercise**)

$$\rho(x) = \frac{1}{\pi}M(x)\sqrt{x^2 + \frac{2t - \sqrt{4t^2 + 48\kappa}}{3\kappa}} \tag{34}$$

$$M(x) = \frac{\kappa}{2}x^2 + \frac{2t + \sqrt{t^2 + 12\kappa}}{6} \ , \tag{35}$$

and one can verify (**exercise**) that $M(x)$ vanishes within the interval of support when $t - 2\sqrt{\kappa}$ and becomes even *negative* for $t < -2\sqrt{\kappa}$. This signals that the interval of support for $t = -2\sqrt{\kappa}$ is about to "split" into two and the *assumption* that the support is only one interval is about to become invalid.



Compendium of the results of DZ analysis

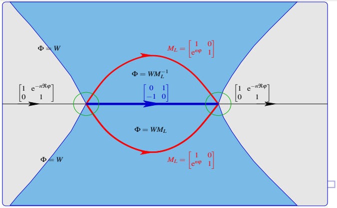
Under the 1–cut assumption we can be rather concise in summarizing the results (for more cuts, one needs to introduce Theta functions on hyperelliptic Riemann surfaces)

Brief summary

$$Y(z;n) = \mathrm{e}^{\frac{n}{2}\ell\sigma_3}\mathscr{E}(z;n)\begin{cases} \Psi_\infty(z)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{n\varphi} & 1 \end{bmatrix} \mathrm{e}^{n(g-\frac{\ell}{2})\sigma_3} & \text{upper lens} \\ \Psi_\infty(z) \quad \mathrm{e}^{n(g-\frac{\ell}{2})\sigma_3} & \text{outside} \\ \Psi_\infty(z)\begin{bmatrix} 1 & 0 \\ -\mathrm{e}^{n\varphi} & 1 \end{bmatrix} \mathrm{e}^{n(g-\frac{\ell}{2})\sigma_3} & \text{lower lens} \end{cases} \tag{36}$$

where $\varphi(z) := V(z) - 2g(z) + \ell$ and Ψ_∞ is explicit and, away from the endpoints is extremely simple:

$$\Psi(z) = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}\left(\frac{z-b}{z-a}\right)^{\frac{1}{4}\sigma_3}\begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \tag{37}$$



Summary of asympotics

Asymptotic on the support (a,b) (uniform on compact subsets of)

$$p_n(x) = \Re\left[\left(\mathrm{e}^{\frac{i\pi}{4}}\left|\frac{x-b}{x-a}\right|^{\frac{1}{4}} + \mathrm{e}^{-\frac{i\pi}{4}}\left|\frac{x-a}{x-b}\right|^{\frac{1}{4}}\right)\mathrm{e}^{in\pi\int_x^b \rho(s)\,\mathrm{d}s}\right] \quad (38)$$

On closed subsets of $\mathbb{C}\setminus[a,b]$ (i.e. **outside the lenses**)

$$p_n(z) = (Y_n)_{11}(z) = \mathrm{e}^{ng(z)}(\mathscr{E}\Psi)_{11} = \Psi_{11}(z)\mathrm{e}^{ng(z)}(1+\mathscr{O}(n^{-1})) \quad (39)$$

$$= \frac{1}{2}\left[\left(\frac{z-b}{z-a}\right)^{\frac{1}{4}} + \left(\frac{z-a}{z-b}\right)^{\frac{1}{4}}\right]\mathrm{e}^{ng(z)} \quad (40)$$

Remark 0.1

Potential theory arguments (without any RHP) can give the following **weak asymptotics** for z outside of the (convex hull of the) support of the equilibrium measure:

$$\lim_{n\to\infty}\frac{1}{n}\ln|p_n(z)| = \Re g(z) \quad (41)$$

In a way, the RHP and the DZ method have been able to turn the weak asymptotic into strong, using the same data.

Application: Random Matrices

Random Matrices: definition and goals

The term is very general and indicates the study of particular *ensembles* of matrices endowed with a *probability measure*. Thus the matrix itself is a *random variable*.

The main objective typically is to study

- the statistical properties of the **spectra** (for square matrices ensembles) or **singular values** (for rectangular ensembles). Thus we need to develop an understanding of the *joint probability distribution functions (jpdf)* of the eigen/singular-values.
- the properties of said statistics when the size of the matrix ensemble tends to infinity (under suitable assumption on the probability measure).
- We only consider a class called "Unitary Ensembles" here.

Let $\mathscr{M}$ be a space of Hermitean matrices ($M = \mathscr{M}^\dagger$) of size $n \times n$:
$\mathscr{M} := \{M \in Mat(n,n;\mathbb{C}),\ M_{ij} = M^\star_{ji}\}$
This is a *vector space* and thus carries a natural Lebesgue measure (invariant by translations) which we shall denote by $\mathrm{d}M$.

$$M_{ab} = X_{ab} + iY_{ab}\ ,\quad X_{ab} = X_{ba}\ ,\quad Y_{ab} = -Y_{ba} \tag{42}$$

$$\dim \mathscr{M} = \frac{n}{2}(n+1) + \frac{n}{2}(n-1) = n^2 \tag{43}$$

$$\mathrm{d}M := \prod_{a=1}^{n} \mathrm{d}X_{aa} \prod_{1\le a<b\le n} \mathrm{d}X_{ab}\,\mathrm{d}Y_{ab} \tag{44}$$

LEMMA 0.1

The Lebesgue measure on $Mat(n,n;\mathbb{C})$ is **invariant under conjugation**: $\mathrm{d}M = \mathrm{d}(CMC^{-1})$.

EXERCISE 0.1

Prove the lemma. Hint: the map is linear and so the Jacobian is certainly constant: show that it is unity.

We recall

THEOREM 0.4

Any Hermitean matrix can be diagonalized by a Unitary matrix $U \in \mathscr{U}(n)$ and its eigenvalues are real

$$\mathscr{U}(n) := \{U \in GL_n(\mathbb{C}) \ , \quad U^\dagger U = UU^\dagger = \mathbf{1}_n\} \tag{45}$$

$$M = U^\dagger X U \ , \quad X = \mathrm{diag}(x_1, x_2, \ldots, x_n) \ , \quad x_j \in \mathbb{R}. \tag{46}$$

REMARK 0.2

The diagonalization is **not unique** even if X is semisimple (i.e. with *distinct eigenvalues*) because we can decide on an ordering of the eigenvalues. In general there are $n!$ distinct diagonalizations. The matrix U can be multiplied on the left by an arbitrary diagonal matrix $D = \mathrm{diag}(\mathrm{e}^{i\theta_1}, \ldots, \mathrm{e}^{i\theta_n})$.

Theorem 0.5

The Lebesgue measure on $\mathscr{M}_{ss}$ can be written as

$$dM = \Delta(X)^2 \prod_{i=1}^{n} dx_i \, dU \ , \qquad \Delta(X) := \prod_{1\leq i<j\leq n} (x_j - x_i) = \det\left[x_a^{b-1}\right]_{1\leq a,b\leq n} \tag{47}$$

The connection to Orthogonal Polynomials (in the simplest incarnation) becomes possible only when the probability measure on $\mathscr{M}$ is of the form

$$d\mu(M) = \frac{1}{Z} e^{-\mathrm{Tr} V(M)} \, dM = \frac{1}{Z} e^{-\sum_{a=1}^{n} V(x_a)} \, dM = \tag{48}$$

We stipulate from now on that this is the choice we are presented with, that is that the reduced jpdf on the eigenvalues is

$$\mu(\vec{x}) = \frac{1}{\mathscr{Z}} \prod_{1\leq a<b\leq n} (x_a - x_b)^2 \prod_{a=1}^{n} e^{-V(x_a)} \, dx_a \tag{49}$$

with $\mathscr{Z}$ the appropriate normalization constant.

LEMMA 0.2

We have

$$\frac{1}{\mathscr{Z}} \prod_{1\le a<b\le n} (x_a - x_b)^2 \prod_{a=1}^{n} \mathrm{e}^{-V(x_a)} = \frac{1}{n!} \det\left[K(x_a, x_b)\right]_{1\le a,b\le n} \tag{50}$$

where

$$K(x,y) = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{j,k=0}^{n-1} x^j y^k [\mathfrak{M}]^{-1}_{jk} = \mathrm{e}^{-\frac{V(x)+V(y)}{2}} \sum_{j=0}^{n-1} \frac{p_j(x)p_j(y)}{\|p_j\|^2} , \tag{51}$$

PROPOSITION 0.2

The Kernel $K(x,y)$ has the following properties

$$\int_{\mathbb{R}} K(x,z)K(z,y)\,\mathrm{d}z = K(z,y) \quad \text{(reproducibility)} \tag{52}$$

$$\int_{\mathbb{R}} K(x,x)\,\mathrm{d}x = n \quad \text{(normalization)} \tag{53}$$

Theorem 0.6

Suppose that a kernel $K(x,y)$ has the properties of **reproducibility** *and* **normalization** *(to n). Then*

$$\text{(a)} \quad \int_{\mathbb{R}} \det[K(x_a,x_b)]_{a,b\leq r}\, \mathrm{d}x_r = (n-r+1)\det[K(x_a,x_b)]_{a,b\leq r-1} \tag{54}$$

$$\text{(b)} \quad \int_{\mathbb{R}^{n-r}} \det[K(x_a,x_b)]_{a,b\leq n}\, \mathrm{d}x_{r+1}\ldots \mathrm{d}x_n = (n-r)!\det[K(x_a,x_b)]_{a,b\leq r-1} \tag{55}$$

Remark 0.3

Dyson's theorem says that the JPDF and *all the marginals* (partial integrations thereof) are in the form of a **determinant** built out of the same kernel ⇒ **determinantal random point fields** [11].

Remark 0.4

The whole statistical information is contained in the Kernel expressed by orthogonal polynomials.

Example 0.4 (Density of eigenvalues)

From the JPDF we integrate all variables except one; this gives the **density** of eigenvalues (i.e. the expected number of eigenvalues in the interval $[x,x+dx]$. According to Dyson's theorem

$$\rho_n(x) = K_n(x,x) = \mathrm{e}^{-V(x)} \sum_{j=0}^{n} \frac{p_j(x)^2}{\|p_j\|^2} \tag{56}$$

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