Orthogonal polynomials, some of their applications and asymptotic analysis

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Abstract The course provides an overview to the theory and applications of orthogonal polynomials (OPs) The undergraduate student most likely encounters OPs when discussing separation of variables in solutions of important PDEs, notably the harmonic oscillator in quantum mechanics (i.e. Hermite polynomials). However, their applications cover a much wider range of topics, whose list includes (but is not limited to): elements of combinatorics, number theory (e.g. the proof that the Euler constant is transcendental); integrable systems (e.g. the Toda lattice equations); stochastic models (random matrices); special equations (Painlevé equations).
The topics covered in the course will be :

- Origins, definitions and fundamental properties.
- Asymptotic analysis for large degrees; elements of nonlinear steepest descent analysis
- Some applications to spectral theory of large random matrices (hopefully with mention of Fredholm determinants and Tracy-Widom distribution, time permitting).

The course is aimed at graduate students (or advanced undergraduate) with a solid grasp of complex analysis (contour integration, conformal properties of holomorphic functions, Cauchy theorem(s)), linear algebra and elementary measure theory.

Introduction

Why study Orthogonal Polynomials (OPs):

1. Special functions: solutions of PDE's by separation of variables e.g. Harmonic Oscillator, Hydrogen Atom.
2. Pure mathematics: theory of approximation of functions, continued fractions (number theory),
3. Applications to special dynamical systems: Toda lattice equations, special solutions of nonlinear ODEs/PDEs.
4. Applications to probabilistic systems: Random Matrices, hopping models on the line
5. Recent applications in Data Science

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First definitions: analytic version

Let $\mathrm{d}\mu(x)$ be a *positive measure* on $\Gamma = \mathbb{R}$ such that

1. $\forall \ell \in \mathbb{N},\ x^\ell \in L^2(\mathbb{R}, \mathrm{d}\mu)$; in particular all the **moments** exist properly:

$$\mu_\ell := \int_{\mathbb{R}} x^\ell \, \mathrm{d}\mu(x)$$

2. the function $F(s) := \int_{-\infty}^{s} \mathrm{d}\mu(x) = \mu\big((-\infty, s]\big)$ has infinite number of points of increase.

Definition 1

The Orthogonal Polynomial System (OPS) for the measure $\mathrm{d}\mu$ is the orthogonal system of $L^2(\mathbb{R}, \mathrm{d}\mu)$ obtained by Gram-Schmidt process starting from the dense system of functions $1, x, x^2, \dots, x^\ell, \dots = \mathbb{C}[x]$ (i.e. a *flag*). By convention we denote the **monic** OPs by $p_n(x) = x^n + \dots$:

$$\int_{\mathbb{R}} p_n(x)\overline{p_m(x)}\, \mathrm{d}\mu(x) = h_n \delta_{nm}.$$

First observations/remarks

- All the polynomials p_n have real coefficients; we can dispose of the conjugation.
- The Gram-Schmidt process depends on the **order** of the initial linearly independent system; the GS process on the sequence $1, x^2, x, x^4, x^3, \dots$. also produces OPs, but not the same! The terminology "Orthogonal Polynomials" almost always refers to the standard flag $1, x, x^2, \dots$.

Algebraic definition: (non)-hermitian OPs

Definition 2

A **moment functional** $\mathcal{L}$ is a linear map $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$

Thus $\mathcal{L}$ is uniquely determined by its **moments**

$$\mathcal{L}(x^j) = \mu_j \ , \quad j = 0, 1, \ldots$$

The integration of the previous slide gives an example:

$$\mathcal{L}(p(x)) = \int p(x) \, \mathrm{d}\mu(x)$$

Definition 3

The (generalized or "non hermitian") OPS associated to $\mathcal{L}$ are **monic** polynomials (if they exist)

$$\{p_n(x) : \deg p_n = n, \quad n = 0, 1, 2, \ldots, \}$$

such that

$$\mathcal{L}(p_n p_m) = \begin{cases} 0 & \text{if } n \neq m \\ h_n \neq 0 & \text{if } n = m \end{cases}$$

Note the absence of conjugation! It is rather a bilinear pairing $< p, q > := \mathcal{L}(pq)$

Exercise 1

Prove the following equivalent characterization (either setup): the sequence $\{p_n(x)\}_{n \in \mathbb{N}}$ of polynomials is orthogonal if and only if $p_n \perp x^\ell, \quad \forall \ell \leqslant n-1$.

Asymptotics of (Orthogonal) polynomials I

In several applications it is important to know the asymptotic behaviour when $n \to \infty$

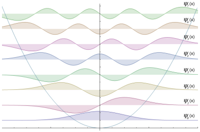
1. For the classical OPs (Hermite/Laguerre/Jacobi) there are explicit integral formulas; this allows easy access to asymptotic behaviour. Important for example in **Quantum Mechanics**:

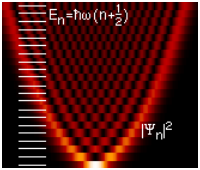
$$-\frac{1}{2}\psi''(x) + \frac{x^2}{2}\psi(x) = \lambda\,\psi(x), \quad \int |\psi(x)|^2\,\mathrm{d}x < +\infty$$

$$\lambda_n := n + \tfrac{1}{2} \qquad \psi_n(x) = \sqrt{\tfrac{1}{2^n n!}}\,\mathrm{e}^{-\frac{x^2}{2}} H_n(x)$$ **Eigenfunctions**

$$H_n(x) = (-1)^n \mathrm{e}^{x^2} \tfrac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-x^2} =$$

$$= \tfrac{n!}{2i\pi} \oint_{|w|=1} w^{-n-1} \mathrm{e}^{2xw - w^2}\,\mathrm{d}w$$ **Hermite polynomials**





Random Matrices

Random Matrix theory is relatively recent, although the roots go back to Wigner in the 50's. The "resurgence" of random matrix theory has been relatively recent also due to outstanding applications to **number theory**, **combinatorics**, **string theory**, **integrable systems**. For example the Gaussian Unitary Ensemble:

$$\mathrm{d}\mu(M) = \frac{1}{\mathcal{Z}_N} \mathrm{e}^{-\mathrm{Tr}\, M^2}\, \mathrm{d}M \ , \quad \mathcal{Z}_N := \int_{\mathcal{H}_N} \mathrm{e}^{-\mathrm{Tr}\, M^2}\, \mathrm{d}M$$

what is the probability that an eigenvalue lies in a certain interval?

Theorem 4 (F. J. Dyson. Correlation between the eigenvalues of a random matrix. *Comm. Math. Phys.*, **19** (1970), 235–250.)

The probability of an eigenvalue λ *of* M *to lie in the interval* $[a, b]$ *is given by*

$$Pr(\lambda \in [a, b]) = \int_a^b \rho_N(\lambda)\, \mathrm{d}\lambda \tag{1}$$

$$\rho_N(\lambda) := \frac{1}{N} \sum_{j=0}^{N-1} \frac{H_j^2(\lambda)}{\sqrt{\pi} 2^j j!} \mathrm{e}^{-\lambda^2} \tag{2}$$

$$K_N(\lambda, \mu) = \frac{H_N(\lambda) H_{N-1}(\mu) - H_N(\mu) H_{N-1}(\lambda)}{2^{N+1} N! (\lambda - \mu)} \tag{3}$$

Example of asymptotic result: Where are the zeroes of the Taylor polynomials of e^z?

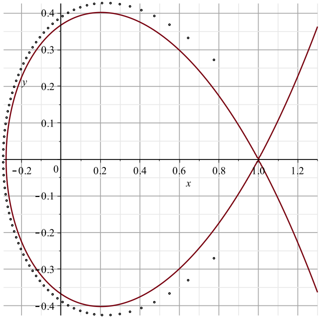
$$\mathrm{e}^z \longrightarrow P_n(z) = \sum_{j=0}^{n} \frac{1}{j!} z^j$$

The exponential function has no zeroes! $P_n(z)$ has n zeroes!. They **fly away** : upon rescaling

$$p_n(z) := P_n(nz)$$

the zeroes of $p_n(z)$ become **dense on the curve**

$$\gamma := \{z : \ |z\mathrm{e}^{1-z}| = 1 \ , \quad |z| \leqslant 1\} \qquad \textbf{(Szegö, 1924)}$$



First properties I

Proposition 1 (Fourier expansion property)

For any polynomial $\pi(x)$ of degree d we have

$$\pi(x) = \sum_{k=0}^{d} c_k p_k(x) \tag{4}$$

$$c_k := \frac{\mathcal{L}(\pi p_k)}{\mathcal{L}(p_k^2)} = \frac{\langle \pi, p_k \rangle}{\langle p_k, p_k \rangle} \tag{5}$$

Proposition 2 (Three-term recurrence relation)

Any OPS satisfies a **three term recurrence relation** *of the form*

$$x p_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \lambda_{n-1} p_{n-1} \ ,$$

Proof: .... ∎

We will see below that the sequence λ_n are nonzero (if the OPS exists)

First properties II

The subspace $\mathbb{P}_N$ of polynomials of degree $\leqslant N-1$ has a dedicated orthogonal projector which is represented by the integral operator:

$$K_N[\pi] := \sum_{j=0}^{n-1} \frac{\langle \pi, p_j \rangle}{\langle p_j, p_j \rangle} p_j(x) = \int \pi(y) \overbrace{\sum_{j=0}^{N-1} \frac{p_j(y)p_j(x)}{h_j}}^{\mathcal{K}_N(x,y)} \, \mathrm{d}\mu(y)$$

A consequence of the three term recurrence relation we have

Theorem 5 (Christoffel-Darboux Identity)

Let $\{p_n\}_\mathbb{N}$ be the (monic) OPs for a moment functional $\mathcal{L}$ so that $\mathcal{L}(p_n^2) = h_n$ Then the following identity holds

$$\mathcal{K}_N(x,y) = \sum_{n=0}^{N-1} \frac{p_n(x)p_n(y)}{h_n} = \frac{1}{h_{N-1}} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x-y} \tag{6}$$

(The expression on the RHS is called the **Bezoutian** *of p_n and p_{n+1})*

Exercise 2

Prove the Christoffel-Darboux Identity using the three term recurrence relation. Hint: multiply both sides by $(x-y)$.

First properties III

Definition 6

The n-th Hankel determinant is defined as

$$\Delta_n := \det \left[\mu_{i+j}\right]_{0 \leqslant i,j \leqslant n-1} = \det \begin{bmatrix} \mu_0 & \mu_1 & \cdots & \mu_{n-1} \\ \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_2 & & & \mu_{n+2} \\ \vdots & & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} \end{bmatrix}$$

Proposition 3

A necessary and sufficient condition for the existence of an OPS is that $\Delta_n \neq 0$, $\forall n \in \mathbb{N}$.

Exercise 3

If $\mathcal{L}$ *is the moment functional of a positive measure (setup 1) then* $\Delta_n > 0$ *for all* n. *Hint: Show that the Hankel matrix is positive definite by integrating* $\int_{\mathbb{R}} \left(\sum_{j=0}^{n} c_j x^j\right)^2 \mathrm{d}\mu(x) > 0$

Construction I

Here they are:

$$p_n(x) := \frac{1}{\Delta_n} \det \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & & \mu_{n+1} \\ \mu_2 & & & & \mu_{n+2} \\ \vdots & & & & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-2} & \mu_{2n-1} \\ 1 & x & x^2 & \cdots & x^n \end{bmatrix}$$

This formula is exact, but it is *not useful* for either numerical or asymptotic analysis (due to typical instability and ill-condition of the matrix of moments).

Problem 7

Let $\{p_n(x)\}$ be a sequence of monic polynomials of degree n ($p_n(x) = x^n + \dots$) satisfying a recurrence relation of the form

$$xp_n(x) = p_{n+1}(x) + \beta_n p_n(x) + \lambda_{n-1} p_{n-1}(x)\ ,\ n = 0, \dots,\quad p_{-1}(x) \equiv 0.$$

where λ_n are (possibly complex) **nonzero** *constants for all $n \in \mathbb{N}$. Show that*

- *two consecutive polynomials cannot share a common root;*
- *the polynomials $p_{n+1}(x)$ can be written as*

$$p_{n+1}(x) = CharPoly \begin{bmatrix} \beta_0 & 1 & & & & \\ \lambda_0 & \beta_1 & 1 & & & \\ & \lambda_1 & \beta_2 & 1 & & \\ & & \lambda_2 & \ddots & \ddots & \\ & & & \ddots & \ddots & 1 \\ & & & & \lambda_{n-1} & \beta_n \end{bmatrix}$$

- *Find the eigenvectors for the above matrix corresponding to each eigenvalue* **[Hint: write the three–term recurrence relation in matrix form]**

Problem 8

Consider the **continued fraction expansion**

$$\frac{q_{n+1}(x)}{p_{n+1}(x)} = \cfrac{-\lambda_0}{x - \beta_0 + \cfrac{-\lambda_1}{x-\beta_1 + \cfrac{-\lambda_2}{\ddots + \cfrac{-\lambda_{n-1}}{x-\beta_{n-1}}}}} \tag{7}$$

and show that the denominators are indeed the OPs.

Problem 9

The following formulas hold (also for the "algebraic version" if we ignore the inequalities in the statements)

$$\int_{\mathbb{R}} p_n^2 \, \mathrm{d}\mu = h_n = \frac{\Delta_{n+1}}{\Delta_n} > 0 \tag{8}$$

$$\lambda_{n-1} = \frac{h_n}{h_{n-1}} = \frac{\Delta_{n+1}\Delta_{n-1}}{\Delta_n^2} > 0 \tag{9}$$

Hint: use the determinantal expression for p_n.

Inverse problems I

As we have seen, any OPS satisfies a three–term recurrence relation; the converse holds true as well as in the following theorem, due to Favard

Theorem 10 (Favard's theorem)

Let $\{\beta_n\}_{n\in\mathbb{N}}$ and $\{\lambda_n\}_{n\in\mathbb{N}}$ be two arbitrary sequences of numbers with $\lambda_n \neq 0$. Let $p_n(x)$ be defined by the recurrence formula

$$\begin{aligned} p_{n+1}(x) = (x-\beta_n)p_n(x) - \lambda_{n-1}p_{n-1}(x) \\ p_{-1} := 0, \qquad p_0 := 1 \;. \end{aligned} \tag{10}$$

Then there exist a unique (up to multiplicative constant) moment functional $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$ such that the p_n's form its OPS. The moment matrix is positive definite if and only if $\beta_n \in \mathbb{R}$ and $\lambda_n > 0$.

For the proof we refer to e.g. [1]

About the moment problem

Suppose that we are given a sequence of (real) numbers μ_k and we are asked:
Question 1 Is there a positive measure $\mathrm{d}\mu$ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^j \,\mathrm{d}\mu(x) = \mu_j\,, \quad j \geqslant 0$$

Question 2 If such measure exists, is it unique?
The union of these questions constitute the *Hamburger moment problem*.
The answer to the first one is in the affirmative as long as all Hankel determinants are strictly positive.

Theorem 11

A positive measure $\mathrm{d}\mu(x)$ exists if and only if $\Delta_n > 0$, $n \in \mathbb{N}$.

The proof (which could be a presentation) is essentially based on Gauss' quadrature formula and Helly's theorems. Note that –quite obviously– we must have $\mu_0 > 0$.
The second question is much more delicate. If the moments satisfy the conditions $\Delta_n > 0$ (and hence a measure exists), in general it is difficult to characterize when such measure is unique.

Gram-Schmidt like you (probably) have not seen it

The "correct" (numerically stable) GS process goes as follows:

1. Construct the **Gram matrix** (matrix of inner products) of your ordered basis:

$$\mathbb{H} = \left[\int_{\mathbb{R}} x^{j+k-2}\,\mathrm{d}\mu(x)\right]_{j,k=1}^{\infty} = \begin{bmatrix} \mu_0 & \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \cdots & \\ \mu_2 & \mu_3 & \cdots & & \\ \mu_3 & \cdots & & & \end{bmatrix}$$

2. Perform LDU decomposition (i.e. Gauss–Jordan elimination, a.k.a., essentially, Cholesky decomposition)

$$\mathbb{H} = LDL^t, \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ \star & 1 & 0 & 0 & \cdots \\ \star & \star & 1 & 0 & \cdots \\ \star & \star & \star & 1 & \cdots \\ & & & & \ddots \end{bmatrix} = \mathbf{1} + N$$

3. [**Exercise**] The OPS can be arranged as an infinite vector $\mathbf{p}(x) := (p_0(x), p_1(x), \ldots, p_n(x), \ldots)^t$ ("wave vector") and

$$\mathbf{p}(x) = L^{-1} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{bmatrix}$$

Some examples: "classical" OPS I

Definition 12

Hermite polynomials The Hermite polynomials are defined by the conditions

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} H_n(x) H_m(x)\, \mathrm{d}x = \sqrt{\pi} 2^n n! \delta_{mn}$$

The **generating function** for Hermite polynomials is defined (formally at first) by

$$G(x;w) := \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} w^n$$

We have immediately

Proposition 4

The generating function for the Hermite polynomials is given by

$$G(x;w) = \mathrm{e}^{2xw-w^2}$$

We then verify the orthogonality. To this end we compute

$$\int_{\mathbb{R}} G(x;w)G(x;z)\mathrm{e}^{-x^2}\,\mathrm{d}x = \int_{\mathbb{R}} \mathrm{e}^{-x^2+2x(w+z)-w^2-z^2}\,\mathrm{d}x = \tag{11}$$

$$= \mathrm{e}^{-w^2-z^2}\int_{\mathbb{R}} \mathrm{e}^{-x^2+2x(w+z)}\,\mathrm{d}x = \tag{12}$$

$$\mathrm{e}^{-w^2-z^2+(z+w)^2}\int_{\mathbb{R}} \mathrm{e}^{-(x-z-w)^2}\,\mathrm{d}x = \mathrm{e}^{2zw}\int_{\mathbb{R}} \mathrm{e}^{-s^2}\,\mathrm{d}s = \sqrt{\pi}\mathrm{e}^{2zw} \tag{13}$$

Writing out the two generating functions(and using dominated convergence to pull out the sum)

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} z^n w^m \int \frac{H_n(x)H_m(x)}{n!m!}\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\sum_{n=0}^{\infty}\frac{2^n z^n w^n}{n!}$$

Comparing the two double series we conclude that

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} H_n(x)H_m(x)\,\mathrm{d}x = \sqrt{\pi}2^n n!\delta_{mn}$$

■

The generaiting function yields more formulas

$$y'' - 2xy' + 2ny = 0 \ , \qquad y = H_n(x) \tag{14}$$

$$\mathrm{e}^{-x^2} H_n(x) = (-1)^n \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \mathrm{e}^{-x^2} \tag{15}$$

$$\frac{H_n(x)}{n!} = \sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu}{\nu!} \frac{(2x)^{n-2\nu}}{(n-2\nu)!} \tag{16}$$

$$\lim_{x\to\infty} x^{-n} H_n(x) = 2^n \tag{17}$$

$$\begin{aligned} H_n(x) &= 2xH_{n-1}(x) - 2(n-1)H_{n-2}(x) \\ H_0 &= 1, \ H_1 = 2x \end{aligned} \tag{18}$$

$$\sum_{\nu=0}^{n} \frac{H_\nu(x)H_\nu(y)}{2^\nu \nu!} = \frac{1}{2^{n+1}n!} \frac{H_{n+1}(x)H_n(y) - H_n(x)H_{n+1}(y)}{x-y} \tag{19}$$

$$H_n'(x) = 2nH_{n-1}(x) \ , \qquad H_n(x) = 2xH_{n-1}(x) - H_{n-1}'(x) \tag{20}$$

$$\sum_{\nu=0}^{n} \binom{n}{\nu} H_\nu(x) H_{n-\nu}(y) = 2^{n/2} H_n(\sqrt{2}(x+y)) \tag{21}$$

$$H_n(x) = \frac{n!}{2i\pi} \oint_{|w|=1} w^{-n-1} \mathrm{e}^{2xw - w^2} \, \mathrm{d}w \tag{22}$$

In particular formula 22 is useful to obtain asymptotics for large n using the standard **steepest descent method**.

These are orthogonal in a "discrete" sense.

$$G(x;w) := \mathrm{e}^{-aw}(1+w)^x = \sum_{m=0}^{\infty} \frac{(-a)^m w^m}{m!} \sum_{n=0}^{\infty} \binom{x}{n} w^n$$

Using the Cauchy product formula for the two series one finds

$$G(x;w) = \sum_{n=0}^{\infty} P_n(x) w^n \ , \quad P_n(x) = \sum_{k=0}^{n} \binom{x}{k} \frac{(-a)^{n-k}}{(n-k)!} \tag{23}$$

Note that $P_n(x)$ are indeed polynomials of degree n since

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}$$

They are called **Charlier polynomials**

$$\sum_{k=0}^{\infty} P_m(k) P_n(k) \frac{a^k}{k!} = \frac{\mathrm{e}^a a^n}{n!} \delta_{nm}$$

This sum can be interpreted as an integral with respect to a measure (up to an overall multiplicative constant) supported on $\mathbb{N}$ and of the form

$$\mathrm{d}\mu(x) = \sum_{n=0}^{\infty} \frac{a^k}{k!} \delta_{x=n}$$

where $\delta_{x=n}$ here means the "Dirac delta" mass distribution.

Laguerre polynomials

They are orthogonal polynomials satisfying

$$\int_0^\infty \mathrm{e}^{-x} x^\alpha L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\,\mathrm{d}x = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{nm} \tag{24}$$

$$L_n^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}\mathrm{e}^x \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(x^{n+\alpha}\mathrm{e}^{-x}\right) \tag{25}$$

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n}\binom{n+\alpha}{n-k}\frac{(-x)^k}{k!} \tag{26}$$

Formula (25) can be reexpressed as

$$L_n^{(\alpha)} = \frac{1}{2i\pi}\oint_{|t|=1/2}\frac{\mathrm{e}^{\frac{xt}{t-1}}}{(1-t)^{\alpha+1}t^{n+1}}\,\mathrm{d}t$$

The generating function is

$$G(x;w) = \frac{1}{(1-w)^{\alpha+1}}\exp\left(\frac{wx}{w-1}\right) \tag{27}$$

Asymptotic methods and questions

Why? How?

Certain questions naturally arise in studying OPS:

1. What is the behaviour of the zeros of the OPS? Are they "predictable" for large degrees? This is also important for numerical applications due to, e.g., the Gauss quadrature methods:

Theorem 13 (Gauss quadrature formula)

There exist (positive) numbers $A_1^{(n)}, \dots, A_n^{(n)}$ such that, for any polynomial $\pi(x)$ of degree $\leqslant 2n-1$ we have

$$\int_{\mathbb{R}} \pi(x)\,\mathrm{d}\mu(x) = \sum_{j=1}^{n} A_j^{(n)} \pi(x_j^{(n)})$$

where $x_j^{(n)}$ are the roots of $p_n(x)$ (OP). The numbers are given by

$$A_j^{(n)} := \int_{\mathbb{R}} \frac{p_n(x)}{(x - x_j^{(n)}) p_n'(x_j^{(n)})}\,\mathrm{d}\mu(x)$$

2. The Hankel determinants themselves Δ_n are of interest in application to the Toda and Kadomtsev-Petviashvili equations, and their large n behaviour.

3. In the application to Random Matrices, this is necessary to address the questions of **universality** (see later).

[See Wikipedia]

$$\int_{\mathbb{R}} \mathrm{e}^{-x^2} H_n(x) H_m(x)\, \mathrm{d}x = \sqrt{\pi} 2^n n! \delta_{mn}$$

1. The **"bulk"** asymptotic: for $|x| << 2n+1$

$$\mathrm{e}^{-\frac{x^2}{2}} H_n(x) \sim \left(\frac{2n}{e}\right)^{\frac{n}{2}} \sqrt{2} \cos\left(x\sqrt{2n+1-\frac{x^2}{3}} - \frac{n\pi}{2}\right)\left(1-\frac{x^2}{2n+1}\right)^{-\frac{1}{4}}$$

2. The **"edge"** asymptotic: for $x = \sqrt{2n+1} + \frac{t}{\sqrt{2}n^{\frac{1}{6}}}$ and $|t|\mathcal{O}(n^{\frac{1}{6}})$

$$\mathrm{e}^{-\frac{x^2}{2}} H_n(x) = \pi^{\frac{1}{4}} 2^{\frac{n}{2}+\frac{1}{4}} \sqrt{n!}\, n^{-\frac{1}{12}} \left(\mathrm{Ai}\,(t) + \mathcal{O}\left(n^{-\frac{2}{3}}\right)\right)$$

where Ai is the Airy function, namely the only solution of

$$f''(t) = t f(t)$$

that is bounded for $t \in \mathbb{R}$ (any other solution is unbounded).

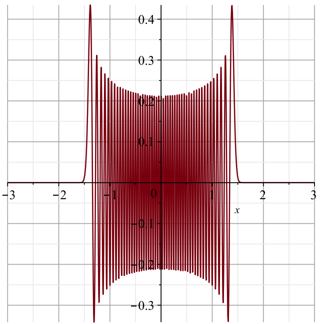
Remark 1

Note that there is a transition of behaviour (in spirit not different from phase transitions in thermodynamics) from the (growing) region $|x| << \sqrt{2n+1}$ and the region $|x| \simeq \sqrt{2n+1}$ (outside it is possible and easier to derive formulas too). For this reason often one studies the OPS in the "rescaled" variable $z = \sqrt{n}x$ so that the "bulk" consists of the interval $(-\sqrt{2}, \sqrt{2})$ and the "edge" becomes

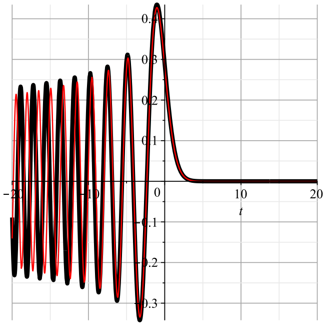
Visual Aid:

Rescaled, orthonormal eigenfunction

$$\psi_n(z) := \frac{1}{\sqrt{h_n}} \mathrm{e}^{-nz^2} H_n(\sqrt{n}z)$$



$\psi_n(z)$ with $n = 100$. Observe the visible envelope of oscillations in the interval $|z| < \sqrt{2}$. Observe the "chirp" and decay outside.



$\psi_n(\sqrt{\frac{2n+1}{n}} + \frac{t}{\sqrt{2}n^{\frac{2}{3}}})$ (black) with $n = 100$ versus $\sqrt{\frac{2}{\pi}}\mathrm{Ai}(t)$ (red). Observe the visible shift of phase as we get further in the bulk.