Groups, rings and the Yang–Baxter equation

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I will discuss some problems related to a discrete version of the Yang–Baxter equation. In particular, I will concentrate on the algebraic structure of skew braces.

Why skew braces?

- The original motivation is the study of set-theoretic solutions to the Yang–Baxter equation (YBE).
- The definition extends that of Rump and is motivated by the work of Cedó, Jespers and Okninski.
- Skew braces put together several ideas that were flying around for years.

A solution (to the YBE) is a pair (X, r), where X is a set and

$$r\colon X \times X \to X \times X, \quad r(x,y) = (\sigma_x(y), \tau_y(x)),$$

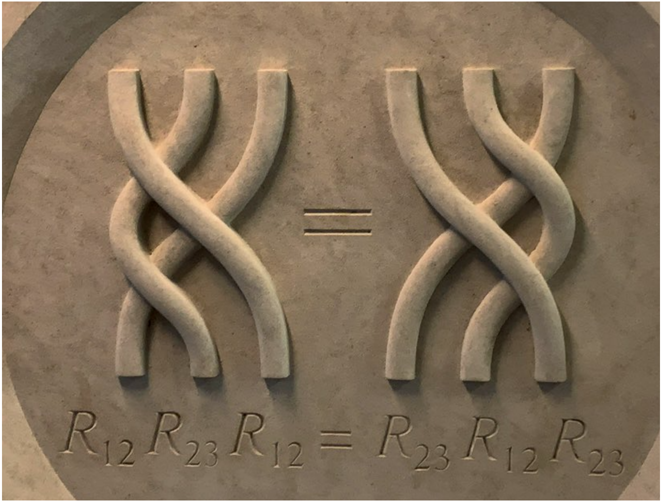
is a bijective map such that

- the maps $\sigma_x\colon X \to X$ are bijective for all $x \in X$,
- the maps $\tau_x\colon X \to X$ are bijective for all $x \in X$, and
- $r_1 r_2 r_1 = r_2 r_1 r_2$, where

$$r_1 = r \times \mathrm{id} \quad \text{and} \quad r_2 = \mathrm{id} \times r.$$

First works: Gateva–Ivanova and Van den Bergh; Etingof, Schedler and Soloviev; Gateva–Ivanova and Majid.

The Yang–Baxter equation



Examples:

- The flip: $r(x, y) = (y, x)$.
- Let X be a set and $\sigma, \tau \colon X \to X$ be bijections such that $\sigma\tau = \tau\sigma$. Then

  $$r(x, y) = (\sigma(y), \tau(x))$$

  is a solution.
- Let $X = \mathbb{Z}/n$. Then

  $$r(x, y) = (2x - y, x) \quad \text{and} \quad r(x, y) = (y - 1, x + 1)$$

  are solutions.

More examples:

If X is a group, then

$$r(x,y) = (xyx^{-1}, x) \quad \text{and} \quad r(x,y) = (xy^{-1}x^{-1}, xy^2)$$

are solutions.

Why?

These solutions...

- have very nice algebraic and combinatorial properties;
- appear in the representation theory of braid groups;
- produce combinatorial invariants of knots;
- motivate challenging problems in other research areas;
- physical applications: Doikou and Smoktunowicz, Gombor and Pozsgay.

We can start with involutive solutions. A solution (X, r) is involutive if $r^2 = \mathrm{id}$.

If (X, r) is involutive, then

$$\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(x)$$

for all $x, y \in X$.

How many solutions are there?

The number of involutive solutions.

n	4	5	6	7	8	9	10
sols	23	88	595	3456	34530	321931	4895272

Solutions of size 9 and 10 were computed with Akgün and Mereb using constraint programming techniques.

Problem

How many involutive solutions (up to isomorphism) of size 11 are there?

More challenging:

Problem

Estimate the number of solutions of size n for $n \to \infty$.

An involutive solution (X,r) is indecomposable if the group

$$\mathcal{G}(X,r)=\langle \sigma_x : x\in X\rangle$$

acts transitively on X.

Problem

Construct indecomposable solutions of small size.

Indecomposable solutions are related to the theory of permutation groups.

More challenging:

Problem

Prove that “almost all” solutions are non-indecomposable.

Let (X, r) be a solution. The structure group of (X, r) is the group $G(X, r)$ with generators X and relations

$$xy = uv$$

whenever $r(x, y) = (u, v)$.

Facts:

- The group $G(X, r)$ acts on X.
- The solution r on X "extends" to a solution on $G(X, r)$.

A concrete example

Let $X = \{1,2,3,4\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$ be the solution given by

$$\begin{array}{llll} \sigma_1 = (12), & \sigma_2 = (1324), & \sigma_3 = (34), & \sigma_4 = (1423), \\ \tau_1 = (14), & \tau_2 = (1243), & \tau_3 = (23), & \tau_4 = (1342). \end{array}$$

The group $G(X,r)$ with generators x_1, x_2, x_3, x_4 and relations

$$\begin{array}{lll} x_1^2 = x_2x_4, & x_1x_3 = x_3x_1, & x_1x_4 = x_4x_3, \\ x_2x_1 = x_3x_2, & x_2^2 = x_4^2, & x_3^2 = x_4x_2. \end{array}$$

A concrete example

The group $G(X,r)$ admits a faithful linear representation inside $\mathbf{GL}_5(\mathbb{Z})$ given by

$$x_1 \mapsto \begin{pmatrix} 0&1&0&0&1\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad x_2 \mapsto \begin{pmatrix} 0&0&0&1&0\\ 0&0&1&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$x_3 \mapsto \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&1\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad x_4 \mapsto \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 1&0&0&0&1\\ 0&0&0&0&1 \end{pmatrix}.$$

A concrete example

Moreover, the map $G(X, r) \to \mathbb{Z}^4$,

$$x_1 \mapsto \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad x_3 \mapsto \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad x_4 \mapsto \begin{pmatrix}0\\0\\0\\1\end{pmatrix},$$

is bijective.

The extra information we have on structure groups is the skew brace structure.

A skew brace is a triple $(A, +, \circ)$, where $(A, +)$ and $(A, \circ)$ are groups and

$$a \circ (b + c) = a \circ b - a + a \circ c$$

holds for all $a, b, c \in A$.

Terminology:

- $(A, +)$ is the additive group of A (even if it is non-abelian) .
- $(A, \circ)$ is the multiplicative group of A.
- A is of abelian type if its additive group is abelian.

Examples:

- Radical rings.
- Trivial skew braces: Any additive group G with $g \circ h = g + h$ for all $g, h \in A$.
- An additive exactly factorizable group G (i.e. $G = A + B$ for disjoint subgroups A and B) is a skew brace with

$$g \circ h = a + h + b,$$

where $g = a + b$, $a \in A$ and $b \in B$.

Skew braces produce solutions.

Theorem (with Guarnieri)

If A is a skew brace, then $r_A\colon A \times A \to A \times A$,

$$r_A(a,b) = (-a + a \circ b, (-a + a \circ b)' \circ a \circ b)$$

is a solution to the YBE.

Here z' denotes the inverse of z with respect to $\circ$.

Applied to the skew brace of a group factorization, the previous formula produces the solutions found by Weinstein and Xu for factorizable Poisson Lie groups.

Let (X, r) be a solution.

Facts:

- $G(X, r)$ is a skew brace (of abelian type if $r^2 = \mathrm{id}$).
- $\mathcal{G}(X, r)$ is a skew brace (of abelian type if $r^2 = \mathrm{id}$).

Theorem (with Smoktunowicz)

Let (X, r) be a solution. Then there exists a unique skew brace structure over $G(X, r)$ such that its associated solution $r_{G(X,r)}$ satisfies

$$r_{G(X,r)}(\iota \times \iota) = (\iota \times \iota)r,$$

where $\iota \colon X \to G(X, r)$ is the canonical map.

The map ι is injective if $r^2 = \mathrm{id}$.

Skew braces have a universal property:

Theorem (with Smoktunowicz)

Let (X,r) be a solution. If B is a skew brace and $f\colon X\to B$ is a map such that

$$(f\times f)r=r_B(f\times f),$$

then there exists a unique homomorphism $\varphi\colon G(X,r)\to B$ of skew braces such that

$$\varphi\iota=f \qquad \text{and} \qquad (\varphi\times\varphi)r_{G(X,r)}=r_B(\varphi\times\varphi).$$

Similar results were found by Etingof, Schedler and Soloviev, Rump, and Lu, Yan and Zhu.

Let us discuss some algebraic problems related to the structure of skew braces.

Etingof, Schedler and Soloviev proved that the multiplicative group of a finite skew brace of abelian type is always solvable.

Question

Is every solvable finite group the multiplicative group of a skew brace of abelian type?

Recall that a solvable group is a group that can be constructed from abelian groups using extensions.

Using ideas of Rump and Lie theory, Bachiller proved that not every finite solvable group is the multiplicative group of a skew brace of abelian type.

Problem

Find a minimal counterexample.

Some comments:

- These problems are discrete analogs of (disproved) a conjecture of Milnor in the theory of flat manifolds.
- Bachiller's result depends on heavy computer calculations.
- We need to study the structure of skew braces where the additive group is a field (i.e. the circle algebras introduced by Catino and Rizzo).

More challenging:

Problem

Which finite solvable groups appear as multiplicative groups of skew braces of abelian type?

Another challenging problem related to solvability is the following conjecture:

Problem (Byott)

Let A be a finite skew brace such that $(A, +)$ is solvable. Is $(A, \circ)$ solvable?

The problem appeared in one of Byott's papers on Hopf–Galois structures. See also Problem 19.91 of *The Kourovka Notebook, by Khukhro and Mazurov*.

Skew braces are ring-like objects. One has an addition

$$(x, y) \mapsto x + y$$

that may be non-commutative, and a “multiplication”

$$(x, y) \mapsto x * y = -x + x \circ y - y$$

which generally is non-associative.

Let p be a prime number and G be a finite p-group. For $k \geq 1$, let

$$G^k = \langle g^k : g \in G \rangle.$$

Then G^k is a normal subgroup of G.

We say that G is powerful if the following conditions hold: if $p > 2$, then G/G^p is abelian; or if $p = 2$, then G/G^4 is abelian.

The notion goes back to Lubotzky and Mann and plays an important role in several areas of group and Lie theory.

A skew brace A is right nilpotent (RP) if $A^{(n)} = \{0\}$ for some n, where $A^{(1)} = A$ and

$$A^{(k+1)} = A^{(k)} * A = \langle x * a : x \in A^{(k)},\, a \in A \rangle_+,$$

and $y * z = -y + y \circ z - z$.

Conjecture (Shalev–Smoktunowicz)

Let p be a prime number and A be a skew brace of abelian type of size p^m. If the multiplicative group of A is powerful, then A is right nilpotent.

Important fact:
Let (X, r) be an involutive solution. For $x, y \in X$ we define

$$x \sim y \Longleftrightarrow \sigma_x = \sigma_y.$$

This equivalence relation induces a solution on $X/{\sim}$,

$$\mathrm{Ret}(X, r) = (X/{\sim}, \overline{r}),$$

the retraction of X.

An involutive solution (X,r) is multipermutation (MP) if there exist $n \geq 1$ such that $|\mathrm{Ret}^n(X,r)| = 1$.

Problem

Prove that "almost all" solutions are MP.

For example, there are 4895272 solutions of size ten and only 28832 are not MP.

Some comments:

- (X,r) is MP $\iff G(X,r)$ is RN $\iff \mathcal{G}(X,r)$ is RN.

Example:
Let $X = \{1,2,3,4,5\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$, where

$$\sigma_1 = \sigma_2 = \sigma_3 = \mathrm{id}, \quad \sigma_4 = (45), \quad \sigma_5 = (23)(45)$$

and

$$\tau_y(x) = \sigma^{-1}_{\sigma_x(y)}(y).$$

Then (X, r) is MP.

The number of (not multipermutation) involutive solutions.

n	4	5	6	7	8	9	10
sols	23	88	595	3456	34530	321931	4895272
not MP	2	4	41	161	2375	16015	28832

Are there easy ways of detecting multipermutation solutions? Yes! There are results related to the permutation group

$$\mathcal{G}(X, r) = \langle \sigma_x : x \in X \rangle$$

of the solution.

Facts

Let (X, r) be a finite and involutive solution.

1. If $\mathcal{G}(X, r)$ is cyclic, then (X, r) is multipermutation.
2. If $\mathcal{G}(X, r)$ is abelian, then (X, r) is multipermutation.
3. If $\mathcal{G}(X, r)$ has abelian Sylow subgroups and has the Sylow tower property, then (X, r) is multipermutation.

(1) was proved by Rump; (2) was proved by Cedó, Jespers and Okniński and independently by Cameron and Gateva–Ivanova; (3) was proved by Ballester–Bolinches, Meng and Romero.

With Bachiller and Cedó we found a characterization of multipermutation solutions in terms of left orderability of groups.

A group G is said to be left orderable if $<$ is a total ordering on G such that the following holds:

$$x < y \implies zx < zy$$

for all $x, y, z \in G$.

Examples:
Torsion-free abelian groups, free groups, braid groups.

Theorem (with Bachiller and Cedó)

Let (X,r) be a finite involutive solution. Then (X,r) is multipermutation if and only if the group $G(X,r)$ is left orderable.

The implication $\implies$ was proved by Jespers and Okniński and independently by Chouraqui.

Corollary (with Acri and Łutowski)

Let (X, r) be a finite involutive solution. If all Sylow subgroups of $\mathcal{G}(X, r)$ are cyclic, then (X, r) is multipermutation.

The following problem was formulated around 80 years ago:

Kaplansky's unit problem

Let G be a torsion-free group. Does the group algebra $\mathbb{C}[G]$ have only trivial units?

Recall that a trivial unit of $\mathbb{C}[G]$ is an element of the form λg, where $\lambda \in \mathbb{C} \setminus \{0\}$ and $g \in G$.

Kaplansky's question has an affirmative answer if G is abelian.

Kaplansky's question has an affirmative answer if G admits a left ordering.

Kaplansky's question has an affirmative solution if G has the so-called unique product property.

A group G has the unique product property if for all finite non-empty subsets A and B of G there exists $x \in G$ that can be written uniquely as $x = ab$ with $a \in A$ and $b \in B$.

Problem

When $G(X, r)$ has the unique product property?

Example (Jespers and Okniński)
Let $X = \{1,2,3,4\}$ and $r(x,y) = (\sigma_x(y), \tau_y(x))$ be the irretractable solution given by

$$\begin{array}{llll} \sigma_1 = (12), & \sigma_2 = (1324), & \sigma_3 = (34), & \sigma_4 = (1423), \\ \tau_1 = (14), & \tau_2 = (1243), & \tau_3 = (23), & \tau_4 = (1342). \end{array}$$

The group $G(X,r)$ with generators x_1, x_2, x_3, x_4 and relations

$$\begin{array}{lll} x_1^2 = x_2x_4, & x_1x_3 = x_3x_1, & x_1x_4 = x_4x_3, \\ x_2x_1 = x_3x_2, & x_2^2 = x_4^2, & x_3^2 = x_4x_2, \end{array}$$

does not have the unique product property.

Let $x = x_1 x_2^{-1}$ and $y = x_1 x_3^{-1}$ and

$$S = \{x^2y, y^2x, xyx^{-1}, (y^2x)^{-1}, (xy)^{-2}, y, (xy)^2x, (xy)^2, (xyx)^{-1}, yxy, y^{-1}, x, xyx, x^{-1}\}.$$

To prove that $G(X, r)$ does not have the unique product property it is enough to prove that each $s \in S^2 = \{s_1s_2 : s_1, s_2 \in S\}$ admits at least two different decompositions of the form $s = ab = uv$ for $a, b, u, v \in S$.

This set S is taken from the work of Promislow.

Our $G(X,r)$ is a finitely presented group. How can we do all these calculations?

We use a faithful linear representation of $G(X,r)$:

$$x_1 \mapsto \begin{pmatrix} 0&1&0&0&1\\ 1&0&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad x_2 \mapsto \begin{pmatrix} 0&0&0&1&0\\ 0&0&1&0&1\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&0&1 \end{pmatrix},$$

$$x_3 \mapsto \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&1\\ 0&0&1&0&0\\ 0&0&0&0&1 \end{pmatrix}, \qquad x_4 \mapsto \begin{pmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0\\ 1&0&0&0&1\\ 0&0&0&0&1 \end{pmatrix}.$$

Theorem (Etingof, Schedler and Soloviev)

Let (X,r) be a finite involutive solution. If $|X|=n$, then $G(X,r) \hookrightarrow \mathbf{GL}_{n+1}(\mathbb{Z})$.

The same trick works for almost all our solutions but there are some open cases!

Example:
Let $X = \{1, \dots, 8\}$ and $r(x, y) = (\sigma_x(y), \tau_y(x))$, where

$$\begin{aligned}
\sigma_1 &= \sigma_2 = (3745), & \tau_1 &= \tau_2 = (3648),\\
\sigma_3 &= \sigma_4 = (1826), & \tau_3 &= \tau_4 = (1527),\\
\sigma_5 &= \sigma_7 = (13872465), & \tau_5 &= \tau_7 = (16542873),\\
\sigma_6 &= \sigma_8 = (17842563), & \tau_6 &= \tau_8 = (13562478).
\end{aligned}$$

Then (X, r) is not a multipermutation solution. Does $G(X, r)$ have the unique product property?

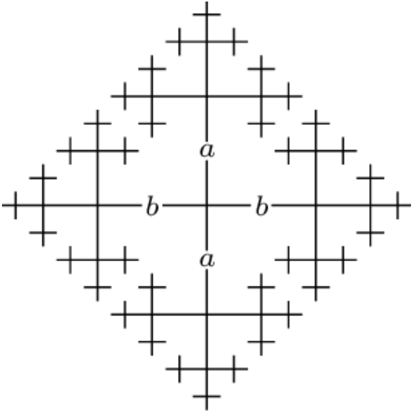
One more problem.

Problem

Let (X, r) be a finite solution. Compute the growth series of $G(X, r)$.

Let G be a group and X be a finite set of generators of G. The Cayley graph of the pair (G, X) is defined as the graph $\Gamma(G, X)$ with vertices in G and edges $G \times X$.

The Cayley graph of the free group in two generators



The ball of radius n is defined as

$$B(1_G, n) = \{g \in G : \mathrm{dist}(1_G, g) \leq n\}$$

and it has size

$$\gamma_{(G,X)}(n) = |B(1_G, n)| < \infty.$$

The pair (G, X) has a rational growth if its growth series

$$\sum_{n=0}^{\infty} \gamma_{(G,X)}(n) t^n \in \mathbb{Z}[[t]]$$

is a rational function, i.e. a function of the form $\frac{p(t)}{q(t)}$ for some polynomials $p(t)$ and $q(t)$.

Theorem (Benson)

If G is virtually abelian (i.e. it has a finite index subgroup that is abelian), then (G, X) has a rational growth for all finite X.

Benson's paper contains an algorithm, but not so easy to carry out. Computing the growth series of structure groups is easy in the case of involutive solutions. What about in the general case?