The Camassa-Holm equation–trente ans après: On the interplay between Approximation Theory, Inverse Problems, and non-smooth Solitons.

Anatol Odzijewicz in Memoriam

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Abstract

It has been 30 years since the derivation of the shallow water equation with peaked solitons by R. Camassa and D. Holm. The Camassa-Holm equation (CH) has become one of the most studied non-linear equations in recent years. This talk reviews the interplay between the mathematics of peakons (non-smooth soliton solutions of CH) and Approximation Theory. I will survey some decisive developments shaping my understanding of peakons and my motivation to study peakon-bearing equations. I will highlight the role of the Padé and Hermite-Padé approximations in the solution of inverse problems intrinsically related to these non-linear wave equations. This talk is partly based on my recent joint work on the beam problem with Richard Beals and peakons with Hans Lundmark and Xiang-ke Chang.

The classical problem of a rotating rigid body

The classical Euler equation of a rotating rigid body:

$$\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \boldsymbol{\omega}$$

where $\mathbf{M}, \boldsymbol{\omega} \in \mathbf{R}^3$ and $M_j = \sum_{k=1}^{3} I_{jk}\omega_k$ (I_{jk} - inertia tensor, symmetric positive definite)
In components

$$\frac{dM_i}{dt} = \sum_{j,k=1}^{3} \epsilon_{ijk} M_j \omega_k$$

ϵ_{ijk} is the Levi-Civita completely skew-symmetric tensor.

Lie algebra interpretation

ϵ_{ijk} defines the structure constants for the Lie algebra $so(3)$

Picture (Poincare-Arnold):

- Let G be a Lie group with the Lie algebra $\mathfrak{g}$. $\mathfrak{g}$ acts on itself via the *adjoint* representation ad.
- This action lifts to the dual $\mathfrak{g}^*$, and one gets the *co-adjoint* representation ad^* on the dual.
- Suppose $\mathfrak{g}$ is equipped with an inner product. The inner product induces an isomorphism $A : \mathfrak{g} \to \mathfrak{g}^*$

Then Euler's equation of the rigid body can be interpreted as

$$\boxed{\frac{dm}{dt} = ad^*_{A^{-1}m} m, \quad m \in \mathfrak{g}^*}$$

For the rigid body: $G = SO(3), A = I$ (inertia tensor)
$\frac{d\mathbf{M}}{dt} = -I^{-1}\mathbf{M} \times \mathbf{M}$

Lie algebra of $\mathit{Diff}^+(S^1)$

The natural geometric way of interpreting elements $u \in \mathrm{diff}(S^1)$ is to view them as vector fields $u\,\partial_x$, and the dual space $\mathrm{diff}^*(S^1)$ as the space of quadratic differentials $\Omega^{\otimes^2}$, with the diffeomorphism-invariant pairing

$$\langle m\,dx^2, u\,\partial_x\rangle = \int_{S^1} mu\,dx.$$

The Lie bracket on $\mathfrak{g} = \mathrm{diff}(S^1)$ is the Lie bracket of vector fields,

$$[u\,\partial_x, v\,\partial_x] = (uv_x - u_x v)\,\partial_x,$$

and hence, if we integrate by parts,

$$\begin{aligned}\langle \mathrm{ad}^*_{u\,\partial_x}(m\,dx^2), v\,\partial_x\rangle &= -\langle m\,dx^2, [u\,\partial_x, v\,\partial_x]\rangle \\ &= -\int_{S^1} m\,(uv_x - u_x v)\,dx \\ &= \int_{S^1} \big((um)_x + u_x m\big)\,v\,dx,\end{aligned}$$

so

$$\boxed{\mathrm{ad}^*_{u\,\partial_x}(m\,dx^2) = \big((um)_x + u_x m\big)\,dx^2}$$

A priori there is of course no relation between m and u. However, if we equip the Lie algebra $\mathrm{diff}(S^1)$ with the H^1 inner product

$$(u\,\partial_x, v\,\partial_x) = \int_{S^1} (uv + u_x v_x)\,dx,$$

then after one integration by parts the inner product can be written

$$(u\,\partial_x, v\,\partial_x) = \int_{S^1} (u - u_{xx})\,v\,dx = \langle Au\,dx^2, v\partial_x\rangle,$$

with $A = 1 - \partial_x^2$. In other words we have a map

$$\boxed{u \mapsto m = Au = u - u_{xx}}$$

Euler's equation on $\mathit{Diff}^+(S^1)$ (G. Misiolek 1998)

$$\boxed{m_t = (mu)_x + u_x m, \qquad \text{where } m = u - u_{xx}.}$$

This equation was proposed earlier by Camassa and Holm in 1993 as a model of one-dimensional dispersive waves in shallow water.

R. Camassa and D. Holm, 1993

In (An integrable shallow water equation with peaked solitons. Phys. Rev. Lett., 71(11):16611664, 1993), the equation (CH_κ)

$$u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$$

was proposed as an integrable model of one-dimensional dispersive waves in shallow water, $u(x,t)$ being the fluid velocity in the x direction. Here κ is a positive physical constant. I will only consider the limiting case with $\kappa = 0$, (CH equation, henceforth)

$$u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}.$$

The CH equation may be written as

$$\boxed{m_t + (um)_x + u_x m = 0, \qquad m = u - u_{xx}},$$

Observe that $(um)_x + u_x m = \mathcal{L}_u m = (\partial_x m + m\partial_x)u$ is a Lie derivative of m.

Camassa-Holm paper

Camassa, Holm derived their equation from the Green-Naghdi equations which is a dimensional/Hamiltonian reduction of Euler's equations for an inviscid, incompressible fluid. By an old result of Arnold, Euler's equations of fluids have the structure identical to the rigid body motion, except that the configuration space is Diff_V (volume preserving diffeomorphisms rather than $\mathrm{SO}(3)$). The CH-structure shows up in one of the Green-Naghdi equations

$$m_t = -(\partial_x m + m\partial_x)\frac{\delta H_{GN}}{\delta m} + \text{corrections.}$$

Eventually, $H_{GN} \to H = \int_{\mathbb{R}} mu\,dx$ where $m = u - u_{xx} + \kappa$, and one obtains CH_κ.

Full CH; $\kappa \neq 0$, Eulerian picture

$\mathrm{so}(3)$ is centrally closed but $\mathrm{diff}(S^1)$ is not. Its central extension is Vir! The CH_κ with nonzero κ is Euler's equation on the dual to Vir.

Table: A comparison between CH and KdV

Lie algebra	**Inner product**	**Euler's equation**
$\mathrm{diff}(S^1)$	$L^2(S^1)$	the inviscid Burgers: $u_t + uu_x = 0$
Vir	$L^2(S^1)$	KdV: $u_t + uu_x + u_{xxx} = 0$
$\mathrm{diff}(S^1)$	$H^1(S^1)$	CH: $u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$
Vir	$H^1(S^1)$	CH_κ : $u_t + 2\kappa u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}$

Lax integrability

For the spinning top we have the equation $\mathbf{M_t} = [\omega, \mathbf{M}]$. One can construct a symmetric matrix J out of the tensor I so that the relation between $\mathbf{M}$ and ω takes the form

$$\mathbf{M} = \omega J + J\omega,$$

Then (Manakov) the Lax equation

$$\frac{d}{dt}(\mathbf{M} + zJ^2) = [\omega + zJ, \mathbf{M} + zJ^2]$$

is equivalent to the Euler's equation of the rigid body. This Lax equation can be viewed as a compatibility condition

$$\begin{aligned} (\mathbf{M} + zJ^2)\Psi &= \lambda\Psi \qquad \text{eigenvalue problem} \\ \Psi_t &= (\omega + zJ)\Psi \qquad \text{deformation} \end{aligned}$$

Lax integrability of the CH equation Consider

$$(-\partial_x^2 + \frac{1}{4})\psi = \frac{\lambda}{2} m\,\psi \qquad \text{eigenvalue problem}$$

$$\psi_t = \frac{1}{2}(\frac{1}{\lambda} + u_x)\psi - (\frac{1}{\lambda} + u)\psi_x \qquad \text{deformation equation}$$

Upshot : both the CH and Euler's equation of the rigid body are Lax integrable.

The CH miracle: peaked solitons (peakons)

CH admits weak solutions (with finite H^1-norm) in the form of peak-shaped travelling waves,

$$u(x,t) = c\, e^{-|x-ct|}, \qquad c \in \mathbb{R},$$

known as *peakons* (peaked solitons), on account of their obviously peaked shape together with the fact that they can also be combined via superposition to form *N-peakon* or *multipeakon* solutions of the form

$$u(x,t) = \sum_{k=1}^{N} m_k(t)\, e^{-|x-x_k(t)|},$$

or, since $m = (1-\partial_x^2)u$

$$\boxed{m(x,t) = 2\sum_{k=1}^{N} m_k(t)\, \delta_{x_k(t)}}$$

The CH turns into a system of ODEs: Peakon equations

$$\dot{x}_k = u(x_k),$$

$$\dot{m}_k = -m_k \langle u_x \rangle (x_k),$$

$$1 \leq k \leq N$$

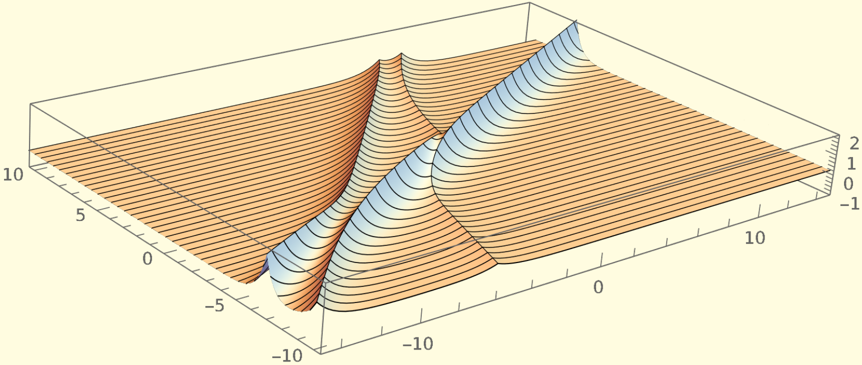


Figure: An example of a three-peakon solution of the Camassa–Holm equation . The graph of $u(x,t) = \sum_{k=1}^{3} m_k(t)\, e^{-|x-x_k(t)|}$ is plotted for $x \in [-15, 15]$ and $t \in [-10, 10]$ In this example, all amplitudes m_k are positive, so it is a *pure peakon solution* (i.e., there are no *antipeakons* with negative m_k).

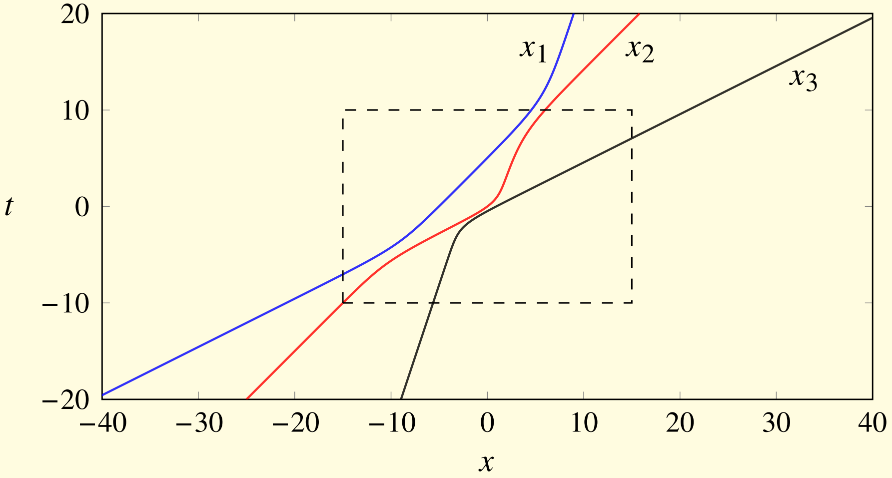


Figure: Positions $x = x_k(t)$ of the three individual peakons in the solution from Figure 1, with the dashed rectangle indicating the region shown there. Note that the ordering $x_1(t) < x_2(t) < x_3(t)$ is preserved for all t, and that the peakons asymptotically (as $t \to \pm\infty$) move in straight lines in the (x, t)-plane, like solitary travelling waves.

The string connection
R. Beals, D.Sattinger, J.S.

To start revealing that connection, (for now t is frozen) we make a Liouville transformation, i.e., a change of dependent and independent variables with the purpose of eliminating the constant term $-\frac{1}{4}$ in the differential operator $\partial_x^2 - \frac{1}{4}$ appearing in the first Lax equation.

$$\left(\partial_x^2 - \tfrac{1}{4}\right)\psi(x) = -\tfrac{1}{2}\lambda\, m(x)\,\psi(x), \qquad x \in \mathbb{R}.$$

The Liouville transformation turns the x-Lax operator into

$$-\partial_y^2\phi(y) = \lambda\, g(y)\, \phi(y), \qquad -1 < y < 1,$$

When $m = 2\sum_{k=1}^{N} m_k \delta_{x_k}$ we obtain the discrete measure g on the interval $(-1, 1)$,

$$g = \sum_{k=1}^{N} g_k\, \delta_{y_k}.$$

This situation corresponds to a *discrete string*:
an idealized object consisting of point masses of weight g_k at the positions y_k, connected by weightless string.

Next we define the *Weyl function* of the discrete string:

$$W(\lambda) = \frac{\phi_y(1;\lambda)}{\lambda\phi(1;\lambda)}.$$

Clearly, this is a rational function with simple poles at the eigenvalues $\lambda_1, \ldots, \lambda_N$ and an extra pole at $\lambda = \lambda_0 = 0$ with residue $W(0) = 1/2 = a_0$; denoting the residues at the other poles by a_k, the partial fractions decomposition of W is

$$W(\lambda) = \frac{1/2}{\lambda} + \sum_{k=1}^{N} \frac{a_k}{\lambda - \lambda_k} = \sum_{k=0}^{N} \frac{a_k}{\lambda - \lambda_k}.$$

In other words,

$$\boxed{W(\lambda) = \int \frac{d\alpha(z)}{\lambda - z}}$$

where the spectral measure is simply $d\alpha = \frac{1}{2}\delta_0 + \sum_{k=1}^{N} a_k \delta_{\lambda_k}$

The following Stieltjes' continued fraction expansion holds:

$$W(\lambda) = \cfrac{1}{\lambda\, l_N + \cfrac{1}{-g_N + \cfrac{1}{\lambda\, l_{N-1} + \cfrac{1}{\ddots + \cfrac{1}{-g_1 + \cfrac{1}{\lambda\, l_0}}}}}},$$

where
l_js are distances between the masses, $l_j = y_{j+1} - y_j$.

We can recover the coefficients of the continued fractions by studying approximations problems. A typical example (the diagonal Padé)

$$\boxed{Q_r(\lambda)W(\lambda) - P_r(\lambda) = O\left(\frac{1}{\lambda^{r+1}}\right)}$$

The polynomials $Q_r(\lambda)$ and $P_r(\lambda)$ (of degree r and $r-1$, respectively, and with $Q(0)=1$) are uniquely determined by this condition. In fact Q_r determines P_r, and Q_r is computable using the moments of the measure $d\alpha$.

The crucial fact: $\{Q_r\}$ are orthogonal polynomials with the respect to the spectral measure $\mathrm{d}\alpha$.

Orthonormal polynomials $\hat{Q}_r$

Let

$$\alpha_n = \int z^n \, d\alpha(z) = \sum_{k=0}^{N} \lambda_k^n \, a_k$$

be the nth moment of the spectral measure $d\alpha$. Then, using the Padé approximation problem, Cramer's rule, and normalizing, we obtain

$$\hat{Q}_r(\lambda) = \frac{\begin{vmatrix} 1 & \lambda & \lambda^2 & \dots & \lambda^r \\ \alpha_0 & \alpha_1 & \alpha_2 & \dots & \alpha_r \\ \alpha_1 & \alpha_2 & \alpha_3 & \dots & \alpha_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{r-1} & \alpha_r & \alpha_{r+1} & \dots & \alpha_{2r-1} \end{vmatrix}}{(\Delta_r^0 \Delta_{r+1}^0)^{\frac{1}{2}}}$$

where

$$\Delta_r^k = \begin{vmatrix} \alpha_k & \alpha_2 & \dots & \alpha_{k+r} \\ \alpha_{k+1} & \alpha_{k+2} & \dots & \alpha_{k+r+1} \\ \vdots & \vdots & \vdots & \vdots \\ \alpha_{k+r} & \alpha_{k+r+1} & \dots & \alpha_{k+2r} \end{vmatrix}$$

Orthonormal polynomials

Why is this interesting?

$$\boxed{I_{N-k} = \hat{Q}_k(0)^2}$$

This fact has significant ramifications for the collisions of peakons! To get there, let us now turn on the time.

CH induced Isospectral Deformation

The CH equation induces an *isospectral deformation* of the string with Dirichlet boundary conditions; as time passes, the mass distribution of the string changes, but its Dirichlet spectrum remains the same. More precisely, if we split

$$\alpha = \frac{1}{2}\delta_0 + \sum_{k=1}^{N} a_k \delta_{\lambda_k}$$

and set

$$\hat{\alpha} := \sum_{k=1}^{N} a_k \delta_{\lambda_k},$$

then the CH flow on the string side reads

$$\boxed{\alpha(t) = \frac{1}{2}\delta_0 + e^{\frac{t}{\lambda}}\hat{\alpha}(0).}$$

Collisions of Peakons

Theorem (BSS)

1. *If the measure m is positive there are no collisions.*
2. *If the measure m is a signed measure peakons may collide.*
3. *The Sobolev norm H^1 of u is preserved through the collision.*

Further comments on collisions

Since at the collision of the k-th peakon at time t_0,

$$0 = l_k(t_0) = \hat{P}_{N-k}(0, t_0),$$

it follows from the Christoffel-Darboux identity that $l_k(t_0)$ and $l_{k+1}(t_0)$ cannot be simultaneously 0. In other words no triple collisions; peakons can only collide in pairs.

This fact appears to be valid for all known to me peakon systems.

Why is CH so special mathematically?

JS:

1. CH is Eulerian and Lax integrable, but so is KdV
2. KdV is an isospectral deformation of the Sturm-Liouville problem in the Schrödinger form; CH is an isospectral deformation of an inhomogenous string. These two are not equivalent!

The connection to the string is, in my opinion, the crux of the matter!

Suppose we are given an arbitrary Hilbert space H (finite dimensional or infinite dimensional) and a self-adjoint operator A with positive, simple spectrum. Then A can be realized as a boundary value problem for an inhomogeneous string. This was proven by M.G. Krein around 1960.

If H is finite dimensional the corresponding string is a discrete string; we are in the peakon sector.

The Peakon Land (Physica D review; joint work with H. Lundmark)

By now (2023) we know a large number of peakon-bearing equations. The most popular, other than the CH, are: the Degasperis-Processi equation

$$m_t + (um)_x + 2u_x m = 0, \qquad m = u - u_{xx}$$

and the V. Novikov equation

$$m_t + ((um)_x + 2u_x m)u = 0, \qquad m = u - u_{xx}.$$

The DP equation (after another Liouville transformation) is an isospectral deformation of the *cubic string*

$$-\partial_y^3\varphi(y) = \lambda g(y)\varphi(y), \qquad \varphi(-1) = \varphi_y(-1) = 0 = \varphi(1).$$

This is a **non-selfadjoint problem**, *but for positive m, and thus g, the spectrum is positive and simple!!!*

There are no collisions.

NV (with H. Lundmark and A. Hone)

NV is an isospectral deformation of the *dual cubic string (after a Liouville transformation)*

$$\frac{\partial}{\partial y}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix} = \begin{bmatrix} 0 & g(y) & 0\\ 0 & 0 & g(y)\\ -\lambda & 0 & 0\end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\\ \varphi_3\end{bmatrix}, \qquad \varphi_2(-1) = \varphi_3(-1) = 0 = \phi_3(1).$$

Again, this is a non-selfadjoint problem, but for positive g the spectrum is positive and simple.

Dual strings

Roughly, if the original discrete string (or discrete cubic string) is characterized by distances $\{l_j\}$ between the masses, and the masses $\{g_j\}$, then for the dual string (dual cubic string) the new distances are given by $\{g_j\}$ and the new masses are $\{l_j\}$. In other words

$$l_j \leftrightarrow g_j$$

In this sense the DP and NV are in **duality** for positive measures (for peakons, no mixed peakons-antipeakons).

Approximation problems

For the cubic string there are **two** Weyl functions $W(\lambda)$ and $Z(\lambda)$, and solving the inverse problem for the cubic string amounts to solving the following **mixed Hermite-Padé** approximation:

$$Q(\lambda)W(\lambda) - P(\lambda) = O(1), \quad Q(\lambda)Z(\lambda) - \widehat{P}(\lambda) = O(1)$$
$$Q(\lambda)Z(-\lambda) - P(\lambda)W(-\lambda) + \widehat{P}(\lambda) = O(\lambda^{-r}),$$
$$\deg Q(\lambda) = \deg \widehat{P}(\lambda) = r - 1,$$
$$P(0) = 1, \widehat{P}(0) = 0.$$

Cauchy biorthogonal polynomials (with M. Bertola and M. Gekhtman)

The solution to these approximation problems can be written in terms of an interesting class of biorthogonal polynomials (Cauchy biorthogonal polynomials)

Definition

Let α and β be two positive measures with support inside $\mathbf{R}_+$. Then the family of biorthogonal polynomials $\{q_n(x), p_n(x), n \in \mathbf{N}\}$ satisfies

$$\langle q_m, p_n \rangle = \int_{\mathbf{R}_+^2} \frac{q_m(x)p_n(y)}{x+y} d\alpha(x) d\beta(y) = \delta_{mn}$$

DP: $\alpha(x) = \delta(x) + \sum_{k=1}^{N} a_k \delta(x - \lambda_k), \quad \beta(x) = x\alpha(x),$
NV: $\alpha(x) = \sum_{k=1}^{N} a_k \delta(x - \lambda_k), \quad \beta(x) = \alpha(x).$

The spectra (in the pure peakon cases) is positive and simple

The cubic string and the dual cubic string are not self-adjoint but still have positive simple spectra. Why?

The cubic string and the dual cubic string are non-selfadjoint oscillatory systems in the sense of Gantmakher and Krein

A beam problem; joint work with R. Beals

$$\boxed{D_x^2[rD_x^2\phi] = \lambda^2\rho\phi, \qquad -1 < x < 1}$$

Lemma (R. Beals and J.S)

Set $\eta = 1/r$. Then the beam problem is equivalent to

$$D_x^2\Phi = \lambda\mathcal{M}\Phi, \qquad \mathcal{M} = \begin{bmatrix} 0 & \eta \\ \rho & 0 \end{bmatrix}, -1 < x < 1$$

The Euler beam is a “string” with an internal structure. (matrix string)

Since the space of initial conditions is 4 dimensional, putting boundary conditions amounts to choosing lower dimensional subspaces of $\mathbb{R}^4$.

Definition (Dirichlet BC)

Let Φ be a 2×2 solution to the matrix string equation such that $\Phi(-1, \lambda) = 0, \Phi_x(-1, \lambda) = \mathbf{1}$. Then the Dirichlet spectrum $\mathcal{S}_{\mathcal{M}} = \{\lambda \in \mathbb{C} : \det \Phi(1, \lambda) = 0\}$.

Isospectral deformations of the DD beam

$$\boxed{\partial_t \Phi = (a + b\partial_x)\Phi}$$

Again, only deformations regular at $\lambda = \infty$ work for measures. The simplest (level 1, only $1/\lambda$ power)

Theorem (R. Beals and J.S)

Let G_D be the Green's function for the DD (Dirichlet) string. Then

$$b(x) = \alpha(x)\mathbf{1} + \frac{2G_D(x,x)}{\lambda}\sigma_1,$$
$$a(x) = \frac{\beta(x)}{2}\sigma_3 - \frac{\alpha_x(x)}{2}\mathbf{1} - \frac{G_{D,x}(x,x)}{\lambda}\sigma_1,$$

$$\alpha(x) = -\int_{-1}^{1} G_D^2(x,\xi)(\rho(\xi)+\eta(\xi))d\xi,$$
$$\beta(x) = \int_{-1}^{1} \mathrm{sign}(x-\xi)G_D(\xi,\xi))(\eta(\xi)-\rho(\xi))d\xi$$

Deformation equations

Recall that $\Phi(x,\lambda)$ satisfies $D_x^2\Phi = \lambda\mathcal{M}\Phi$ where $\Phi(-1,\lambda)=0, \Phi'(-1,\lambda)=\mathbf{1}$ and $\mathcal{M} = \begin{bmatrix} 0 & \eta \\ \rho & 0 \end{bmatrix}$.

Then the isospectral, level one, evolution equations for the DD beam are

$$\boxed{\eta_t = (\alpha\eta)_x + \alpha_x\eta + \beta\eta, \quad \rho_t = (\alpha\rho)_x + \alpha_x\rho - \beta\rho}$$

It is instructive to see how these equations look if the interval $[-1,1]$ is mapped to $\mathbb{R}$:

$$\rho \to m, \quad \eta \to n, \quad \alpha \to u, \quad \beta \to v, \quad \mathcal{M} \to M = \begin{bmatrix} 0 & n \\ m & 0 \end{bmatrix}$$

- $D_x^2 \Phi = \big(\mathbf{1} + \lambda M\big)\Phi$
- $n_t = (un)_x + u_x n + vn, \quad m_t = (um)_x + u_x m - vm$
- $v_x = (m-n), \quad u - u_{xx} = m + n$
- Peakon equations:

$$\begin{aligned} \dot{x}_i &= u(x_i), \\ \dot{m}_j &= -m_j \big\langle u_x - v \big\rangle (x_i) \\ \dot{n}_j &= -n_j \big\langle u_x + v \big\rangle (x_i) \end{aligned}$$

Stieltjes and strings

For the string, the resolvent (the Weyl function)

$$\boxed{W(z) = \cfrac{1}{l_d z + \cfrac{1}{m_d + \cfrac{1}{\ddots + \frac{1}{l_0 z}}}}}$$

Stieltjes and beams

For the beam, the Weyl function $W(\lambda) = \frac{1}{\lambda}\Phi_x(1,\lambda)\Phi(1,\lambda)^{-1}$ (a 2×2 matrix)

$$\boxed{W(\lambda) = \cfrac{1}{l_d \mathbf{1}\lambda + \cfrac{1}{\mathcal{M}_d + \cfrac{1}{\ddots + \frac{1}{l_0 \mathbf{1}\lambda}}}}}$$

Surprise!

Let us go back to the finite interval.

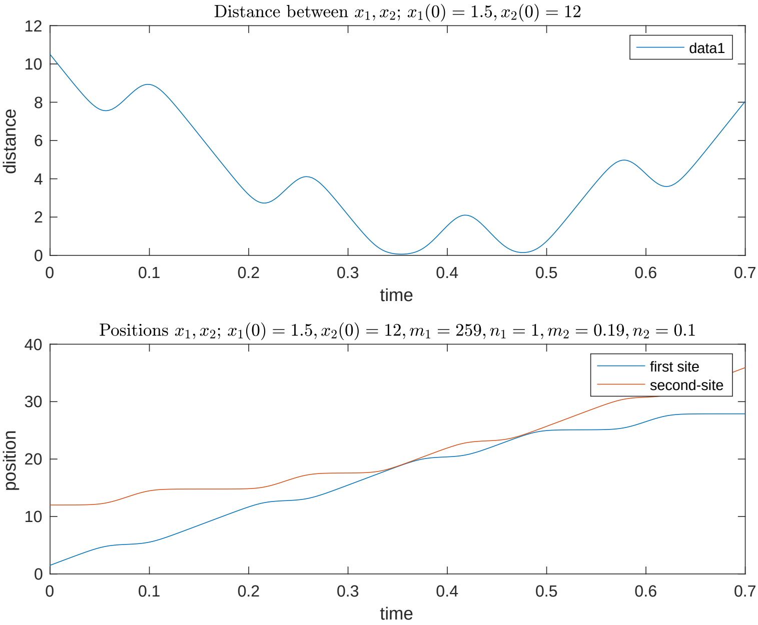
- $\det\big(\Phi(1; z)\big)$ is invariant under the DD beam deformations.
- Surprisingly, so is $\mathrm{Tr}\big(\Phi(1; z)\sigma_1\big)$.

Forthcoming Attractions

Theorem (R. Beals and J.S.)

Let $\mathcal{M} = \sum_{j=1}^{d} \mathcal{M}_j \delta_{x_j}$ *be a discrete, finite, measure,* $L = \Phi(1;\lambda)\sigma_1$*, and let* X *be the hyperelliptic curve whose affine part is* $\det\left(w\mathbf{1} - L\right) = 0$*. Then*

- X *is invariant under the DD, level one, beam flow*
- *the DD beam flow linearizes on the* $Jac(X)$*.*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data1
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Positions x_1, x_2; $x_1(0) = 1.5, x_2(0) = 12, m_1 = 259, n_1 = 1, m_2 = 0.19, n_2 = 0.1$
first site
second-site
position
time

Thank you !

Hope to be back in Bialowieża one day!