Stability Theorem for $\mathbb{Z}_2^n$-Lie supergroups

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Introduction

Existing a unique unitary representation as an extension of a pre-representation is called Stability Theorem. In 2012, Stephane, Neeb and Salmasian [4] prove that Stability Theorem holds in a Banach-Lie group. In 2013, Neeb and Salmasian [3] state that this theorem holds in each Lie supergroup. In 2017, Mohammadi and Salmasian [2] assert that under an extra condition, this theorem holds in a $\mathbb{Z}_2^n$-Lie supergroup. It is worth mentioning that in our work, Stability Theorem is proved unconditionally. Our proof differs from the one in [2]. You can see the concept of $\mathbb{Z}_2^n$-supergeometry in [1, 2].

$\mathbb{Z}_2^n$-vector superspace

A direct sum $V = \bigoplus_{\gamma \in \mathbb{Z}_2^n} V^\gamma$ is called a $\mathbb{Z}_2^n$-vector superspace where V^γ are vector spaces over a commutative field $\mathbb{K}$ with characteristic 0 for every $\gamma \in \mathbb{Z}_2^n$.

$\mathbb{Z}_2^n$-superalgebra

A $\mathbb{Z}_2^n$-vector superspace $V = \bigoplus_{\gamma \in \mathbb{Z}_2^n} V^\gamma$ is called a $\mathbb{Z}_2^n$-superalgebra if there exists an operation of multiplication on V such that $V^\gamma V^\eta \subset V^{\gamma+\eta}$.

- An element $x \in V^{\gamma}$ is called a $\mathbb{Z}_2^n$-homogeneous element of degree or weight $\tilde{x} := \gamma$.
- V is called a commutative algebra if for $\mathbb{Z}_2^n$-homogeneous elements $x, y \in V$ of degree a and b respectively, we have

$$xy = \mathcal{B}(a, b)yx.$$

$\mathcal{B}(a, b) = (-1)^{\langle a,b \rangle}$ where $\langle a, b \rangle = a_1 b_1 + a_2 b_2 + ... + a_n b_n$.

- A $\mathbb{Z}_2^n$-homogeneous element $x \in V^{\gamma}$ is called odd if $p(\tilde{x}) = p(\gamma) := \mathcal{B}(\gamma, \gamma) = -1$.
- A $\mathbb{Z}_2^n$-homogeneous element $x \in V^{\gamma}$ is called even if $p(\tilde{x}) = p(\gamma) := \mathcal{B}(\gamma, \gamma) = 1$.

$\mathbb{Z}_2^n$-Lie superalgebra

A $\mathbb{Z}_2^n$-vector superspace $\mathfrak{g} = \bigoplus_{a \in \mathbb{Z}_2^n} \mathfrak{g}_a$ is caled $\mathbb{Z}_2^n$-Lie superalgebra if there exists $\mathbb{Z}_2^n$-superbracket $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ such that

i) $[.,.]$ is bilinear and for every $a, b \in \mathbb{Z}_2^n$, we have $[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b}$,

ii) for every $x \in \mathfrak{g}_a$ and $y \in \mathfrak{g}_b$ where $a, b \in \mathbb{Z}_2^n$

$$[x, y] = -\mathcal{B}(a, b)[y, x] \ \ (Graded\ \ skew - symmetry),$$

iii) for every $x \in \mathfrak{g}_a$, $y \in \mathfrak{g}_b$ and $z \in \mathfrak{g}_c$ where $a, b, c \in \mathbb{Z}_2^n$ we have

$$[x, [y, z]] = [[x, y], z] + \mathcal{B}(a, b)\mathcal{B}(a, c)[y, [z, x]]$$
$$(Graded\ \ Jacoby\ \ identity).$$

$\mathbb{Z}_2^n$-inner product

Let $\mathcal{H}$ be a $\mathbb{Z}_2^n$-vector superspace. The complex-valued map

$$\langle , \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

is called a $\mathbb{Z}_2^n$-inner product on $\mathcal{H}$ if the following holds:

i) $\langle \lambda u + \mu v, w\rangle = \lambda\langle u, w\rangle + \mu\langle v, w\rangle$, for $\lambda, \mu \in \mathbb{C}$ and $u, v, w \in \mathcal{H}_a$ where $a \in \mathbb{Z}_2^n$ (Linear in the first argument);

ii) $\langle w, v\rangle = \mathcal{B}(a, a)\overline{\langle v, w\rangle}$ for $v, w \in \mathcal{H}_a$ where $a \in \mathbb{Z}_2^n$ (Hermitian symmetric);

iii) $\alpha(a)\langle v, v\rangle \geq 0$, for $v \in \mathcal{H}_a$ where $a \in \mathbb{Z}_2^n$ (nonnegative)

$$\alpha(a) = e^{i\frac{\pi}{2}\mathbf{u}(a)}, where \ \ \mathbf{u}(a) = |\{1 \leq j \leq n \ \ : \ \ a_j = \bar{1}\}|$$

iv) $\langle v, w\rangle = 0$, for $v \in \mathcal{H}_a$ and $w \in \mathcal{H}_b$, $a \neq b$ where $a, b \in \mathbb{Z}_2^n$.

The $\mathbb{Z}_2^n$-vector superspace $\mathcal{H}$ with a $\mathbb{Z}_2^n$-inner product $\langle , \rangle$ is called a $\mathbb{Z}_2^n$-inner product space or $\mathbb{Z}_2^n$-pre-Hilbert superspace.

Let $\langle , \rangle$ be a $\mathbb{Z}_2^n$-inner product on $\mathcal{H}$ and let $v, w \in \mathcal{H}_a$ where $a \in \mathbb{Z}_2^n$. An inner product (in the ordinary sense) on $\mathcal{H}$ is defined as fallows

$$(v, w) = \alpha(a)\langle v, w\rangle$$

The vector space $\mathcal{H}$ with inner product $(,)$ is a pre-Hilbert space in the usual sense. We now define a norm on $\mathcal{H}$ as fallows

$$||v|| = \sqrt{(v, v)} \tag{1}$$

The vector space $\mathcal{H}$ with this norm is a normed linear space.

$\mathbb{Z}_2^n$-Hilbert superspace

A $\mathbb{Z}_2^n$-Hilbert superspace is a complete $\mathbb{Z}_2^n$-pre-Hilbert superspace with norm defined in (1).

definition of adjoint $T^\dagger$

Let $\mathcal{H}$ be a $\mathbb{Z}_2^n$-inner product space with product $\langle , \rangle$ and let $T : \mathcal{H} \to \mathcal{H}$ is a $\mathbb{C}$-linear map of degree a where $a \in \mathbb{Z}_2^n$. The adjoint operator of T is denoted by $T^\dagger$ and defined as follows

$$\langle v, Tw\rangle = \mathcal{B}(a,b)\langle T^\dagger v, w\rangle \qquad for \;\; v \in \mathcal{H}_b$$

definition of adjoint T^*

We now define adjoint T^* with respect to ordinary inner product $(,)$ on $\mathbb{Z}_2^n$-inner product space $\mathcal{H}$ as fallows

$$(Tv, w) = (v, T^* w) \qquad for \;\; every \;\; v, w \in \mathcal{H}$$

For $\mathbb{C}$-linear map $T : \mathcal{H} \to \mathcal{H}$ of degree $a \in \mathbb{Z}_2^n$, we have

$$T^* = \overline{\alpha(a)} T^\dagger. \tag{2}$$

It fallows that for $\mathbb{C}$-linear maps $T, S : \mathcal{H} \to \mathcal{H}$ of degrees $a, b \in \mathbb{Z}_2^n$ respectively, we have

$$T^{\dagger\dagger} = T, \qquad (ST)^\dagger = \mathcal{B}(b, a) T^\dagger S^\dagger.$$

By a $\mathbb{Z}_2^n$-Lie supergroup we mean a Harish-Chandra pair $(G_0, \mathfrak{g}_{\mathbb{C}})$ where G_0 is a common Lie Group and $\mathfrak{g}_{\mathbb{C}}$ is a $\mathbb{Z}_2^n$-graded Lie superalgebra such that there exists an action $Ad : G_0 \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ preserves the $\mathbb{Z}_2^n$-grading and $\mathrm{Ad}|_{\mathfrak{g}_{\mathfrak{o}}} : G_0 \times \mathfrak{g}_{\mathfrak{o}} \to \mathfrak{g}_{\mathfrak{o}}$ is the adjoint action of G_0 on $\mathfrak{g}_{\mathfrak{o}} \cong Lie(G_0)$.[2]

pre–representation

[2] Let $(G_0, \mathfrak{g}_{\mathbb{C}})$ be a $\mathbb{Z}_2^n$-Lie supergroup. A 4-tuple $(\pi, \mathcal{H}, \mathfrak{B}, \rho^{\mathfrak{B}})$ is a pre-representation of $(G_0, \mathfrak{g}_{\mathbb{C}})$ if the following holds

i) $(\pi, \mathcal{H})$ is a smooth unitary representation of the Lie group G_0 on the $\mathbb{Z}_2^n$-graded Hilbert space $\mathcal{H}$ such that $\pi(g)$ preserves the $\mathbb{Z}_2^n$-grading for every $g \in G_0$.

ii) $\mathfrak{B}$ is a dense, G_0-invariant, $\mathbb{Z}_2^n$-graded subspace of $\mathcal{H}$ such that

$$\mathfrak{B} \subseteq \bigcap_{x \in \mathfrak{g}_0} \mathcal{D}_x$$

where $\mathcal{D}_x := \{v \in \mathcal{H} : \frac{d}{dt}|_{t=0} \pi(exp_{G_0}(tx))v \ \ exists\}$ is the domain of the infinitesimal generator $\overline{\mathrm{d}\pi}(x)$ of the one-parameter group $t \mapsto \pi(exp_{G_0}(tx))$. Thus $\overline{\mathrm{d}\pi}(x)v := \frac{d}{dt}|_{t=0} \pi(exp(tx))v$ and $\mathcal{D}(\overline{\mathrm{d}\pi}(x)) = \mathcal{D}_x$.

pre–representation

iii) $\rho^{\mathfrak{B}} : \mathfrak{g}_{\mathbb{C}} \to End_{\mathbb{C}}(\mathfrak{B})$ is a representation of the $\mathbb{Z}_2^n$-Lie superalgebra $\mathfrak{g}_{\mathbb{C}}$, i.e., for x and y of degree a, b respectively, we have

$$\rho^{\mathfrak{B}}([x,y]) = \rho^{\mathfrak{B}}(x)\rho^{\mathfrak{B}}(y) - \mathcal{B}(a,b)\rho^{\mathfrak{B}}(y)\rho^{\mathfrak{B}}(x).$$

iv) For every $x \in \mathfrak{g}_0$, $\rho^{\mathfrak{B}}(x) = \overline{\mathrm{d}\pi}(x)|_{\mathfrak{B}}$ and $\rho^{\mathfrak{B}}(x)$ is essentially skew adjoint.

v) For every $x \in \mathfrak{g}_{\mathbb{C}}$, $\rho^{\mathfrak{B}}(x)^{\dagger} = -\rho^{\mathfrak{B}}(x)$, i.e., for x of degree a, $i\overline{\alpha(a)}^{\frac{1}{2}}\rho^{\mathfrak{B}}(x)$ is a symmetric operator, i.e., $-\overline{\alpha(a)}\rho^{\mathfrak{B}}(x) \subseteq \rho^{\mathfrak{B}}(x)^{*}$.

vi) $\rho^{\mathfrak{B}}$ is a homomorphism of G_0-modulus, i.e., for every $x \in \mathfrak{g}_{\mathbb{C}}$ and $g \in G_0$, $\pi(g)\rho^{\mathfrak{B}}(x)\pi(g)^{-1} = \rho^{\mathfrak{B}}(\mathrm{Ad}(g)x)$.

representation

[2] Let $(G_0, \mathfrak{g}_{\mathbb{C}})$ be a $\mathbb{Z}_2^n$-Lie supergroup. A smooth unitary representation of $(G_0, \mathfrak{g}_{\mathbb{C}})$ is a triple $(\pi, \rho^\pi, \mathcal{H})$ with the following properties.

i) $(\pi, \mathcal{H})$ is a smooth unitary representation of the Lie group G_0 on the $\mathbb{Z}_2^n$-graded Hilbert space $\mathcal{H}$ such that $\pi(g)$ preserves the $\mathbb{Z}_2^n$-grading for every $g \in G_0$.

ii) For the space of smooth vectors; $\mathcal{H}^\infty$, $\rho^\pi : \mathfrak{g}_{\mathbb{C}} \to \mathrm{E}nd_{\mathbb{C}}(\mathcal{H}^\infty)$ is a representation of the $\mathbb{Z}_2^n$-Lie superalgebra $\mathfrak{g}_{\mathbb{C}}$, i.e., for x and y of degrees a, b respectively, one has

$$\rho^\pi([x, y]) = \rho^\pi(x)\rho^\pi(y) - \mathcal{B}(a, b)\rho^\pi(y)\rho^\pi(x).$$

representation

iii) For every $x \in \mathfrak{g}_0$, $\rho^\pi(x) = \overline{\mathrm{d}\pi}(x)|_{\mathcal{H}^\infty}$.

iv) $\rho^\pi(x)^\dagger = -\rho^\pi(x)$ for every $x \in \mathfrak{g}_\mathbb{C}$, i.e., $i\overline{\alpha(a)}^{\frac{1}{2}}\rho^\pi(x)$ is a symmetric operator, i.e., $-\overline{\alpha(a)}\rho^\pi(x) \subseteq \rho^\pi(x)^*$ for x of degree a.

v) ρ^π is a homomorphism of G_0-modulus, i.e.,
$\pi(g)\rho^\pi(x)\pi(g)^{-1} = \rho^\pi(\mathrm{Ad}(g)x)$ for every $x \in \mathfrak{g}_\mathbb{C}$ and $g \in G_0$.

Stability Theorem

The following auxiliary lemma is needed. See [4, Lemma 2.5] for a proof.

lemma

Let P_1 and P_2 be two symmetric operators on a complex Hilbert space $\mathcal{H}$ such that $\mathcal{D}(P_1) = \mathcal{D}(P_2)$. Let $L \subseteq \mathcal{D}(P_1)$ be a dense linear subspace of $\mathcal{H}$ such that $P_1|_L = P_2|_L$. Assume that $P_1|_L$ is essentially self-adjoint. Then $P_1 = P_2$.

Stability Theorem

Let $(\pi, \mathcal{H}, \mathfrak{B}, \rho^{\mathfrak{B}})$ be a pre-representation of a $\mathbb{Z}_2^n$-Lie supergroup $(G_0, \mathfrak{g}_{\mathbb{C}})$. Then there exists a unique smooth unitary representation of $(G_0, \mathfrak{g}_{\mathbb{C}})$, say $(\pi, \rho^{\pi}, \mathcal{H})$, such that $\rho^{\pi}(x)|_{\mathfrak{B}} = \rho^{\mathfrak{B}}(x)$.

proof

To prove the existence of ρ^{π}, we set $\rho^{\pi}(x) = \overline{\rho^{\mathfrak{B}}(x)}$ for every $x \in \mathfrak{g}_{\mathbb{C}}$, where $\overline{\rho^{\mathfrak{B}}(x)}$ is a smallest closed extension on $\mathcal{H}$. First we show that if $v \in \mathcal{D}^{\infty}$ then $\overline{\rho^{\mathfrak{B}}(x)}v$ belongs to $\mathcal{D}^{\infty}$, where $D^{\infty} := \bigcap\limits_{n \in \mathbb{N}} D^n$ and $D^1 := \bigcap\limits_{x \in \mathfrak{g}_{\mathfrak{o}}} D_x$ and for $n > 1$

$$D^n := \{v \in D^1 : \overline{\mathrm{d}\pi}(x)v \in D^{n-1} \quad \forall x \in \mathfrak{g}_{\mathfrak{o}}\}.$$

Step 1

It is sufficient to prove that $\overline{\rho^{\mathfrak{B}}(x)}v \in \mathcal{D}^n$ for every $x \in \mathfrak{g}_{\mathbb{C}}$ of degree a, $v \in \mathcal{D}^{n+1}$ and $n \in \mathbb{N}$. We do this by induction on n. For $y \in \mathfrak{g}_0$, $w \in \mathfrak{B}$ and $v \in \mathcal{D}^2$, we have

$$(\overline{\rho^{\mathfrak{B}}(x)}v, \overline{\mathrm{d}\pi}(y)w) = (\overline{\rho^{\mathfrak{B}}(x)}\,\overline{\mathrm{d}\pi}(y)v, w) + (\overline{\rho^{\mathfrak{B}}([x,y])}v, w)$$

It follows that $\overline{\rho^{\mathfrak{B}}(x)}v \in \mathcal{D}((\overline{\mathrm{d}\pi}(y)|_{\mathfrak{B}})^*)$. Since $(\overline{\mathrm{d}\pi}(y)|_{\mathfrak{B}})^* = (\rho^{\mathfrak{B}}(y))^* = -\overline{\rho^{\mathfrak{B}}(y)} = -\overline{d\pi}(y)$ then $\overline{\rho^{\mathfrak{B}}(x)}v \in \mathcal{D}(\overline{\mathrm{d}\pi}(y))$.

By induction hypothesis, for $x_1, ..., x_n \in \mathfrak{g}_0$ and, $v \in D^{n+1}$ we obtain

$$(\overline{\mathrm{d}\pi}(x_{n-1})...\overline{\mathrm{d}\pi}(x_1)\overline{\rho^{\mathfrak{B}}(x)}v, \overline{\mathrm{d}\pi}(x_n)w) = (\overline{\mathrm{d}\pi}(x_n)...\overline{\mathrm{d}\pi}(x_2)\overline{\rho^{\mathfrak{B}}([x, x_1])}v, w) + (\overline{\mathrm{d}\pi}(x_n)...\overline{\mathrm{d}\pi}(x_2)\overline{\rho^{\mathfrak{B}}(x)}\overline{\mathrm{d}\pi}(x_1)v, w)$$

A similar computation shows that $\overline{\mathrm{d}\pi}(x_{n-1})...\overline{\mathrm{d}\pi}(x_1)\overline{\rho^{\mathfrak{B}}(x)}v \in D(\overline{\mathrm{d}\pi}(x_n))$. Consequently, $\overline{\rho^{\mathfrak{B}}(x)}v \in \mathcal{D}^n$.

Since G_0 ia a finite-dimensional Lie group and the representation is smooth, one has $\mathcal{D}^\infty = \mathcal{H}^\infty$[3, Thm 5.3].

Step 2

We show that $\rho^{\pi} : \mathfrak{g}_{\mathbb{C}} \to \mathrm{E}nd_{\mathbb{C}}(\mathcal{H}^{\infty})$ is a representation of the $\mathbb{Z}_2^n$-Lie superalgebra $\mathfrak{g}_{\mathbb{C}}$. For $x \in \mathfrak{g}_{\mathbb{C}}$ of degree a, $v \in \mathcal{D}^{\infty}$ and $c \in \mathbb{R}$ if we set

$$L = \mathfrak{B}, \qquad P_1 v = i\overline{\alpha(a)}^{\frac{1}{2}}\overline{\rho^{\mathfrak{B}}(cx)}v \qquad \text{and} \qquad P_2 = i\overline{\alpha(a)}^{\frac{1}{2}}c\overline{\rho^{\mathfrak{B}}(x)}v$$

by Lemma 15, $P_1 = P_2$ for every $v \in \mathcal{D}^{\infty}$.

For $x, y \in \mathfrak{g}_{\mathbb{C}}$, both of degree a, and $v \in \mathcal{D}^\infty$, by apply Lemma 15, a similar reasoning show that,

$$\overline{\rho^{\mathfrak{B}}(x+y)}v = \overline{\rho^{\mathfrak{B}}(x)}v + \overline{\rho^{\mathfrak{B}}(y)}v.$$

We set

$$L = \mathfrak{B}, \quad P_1 v = i\overline{\alpha(a)}^{\frac{1}{2}}\overline{\rho^{\mathfrak{B}}(x+y)}v, \quad P_2 v = i\overline{\alpha(a)}^{\frac{1}{2}}\left(\overline{\rho^{\mathfrak{B}}(x)} + \overline{\rho^{\mathfrak{B}}(y)}\right)v$$

Now, in order to prove $\rho^{\mathfrak{B}}$ preserves the Lie braket, let $x, y \in \mathfrak{g}_{\mathbb{C}}$ be of degree a, b, respectively. we define two operators T_1 and T_2 with domains $\mathcal{D}(T_1) = \mathcal{D}(T_2) = \mathcal{D}^\infty$ as follows. For $v \in \mathcal{D}^\infty$ we set

$$T_1 v = i(\mathcal{B}(a,b)\overline{\alpha(a)\alpha(b)})^{\frac{1}{2}}\overline{\rho^{\mathfrak{B}}([x,y])}v,$$

$$T_2 v = i(\mathcal{B}(a,b)\overline{\alpha(a)\alpha(b)})^{\frac{1}{2}}(\overline{\rho^{\mathfrak{B}}(x)}\ \overline{\rho^{\mathfrak{B}}(y)}v - \mathcal{B}(a,b)\overline{\rho^{\mathfrak{B}}(y)}\ \overline{\rho^{\mathfrak{B}}(x)}v)$$

where let $x, y \in \mathfrak{g}_{\mathbb{C}}$ be of degree a, b, respectively.

Step 3

To prove ρ^π is a homomorphism of G_0-modulus, we apply Lemma 15 with

$$L = \mathfrak{B}, \qquad P_1 = i\overline{\alpha(a)}^{\frac{1}{2}}\pi(g)\overline{\rho^{\mathfrak{B}}(x)}\pi(g)^{-1}, \quad P_2 = i\overline{\alpha(a)}^{\frac{1}{2}}\overline{\rho^{\mathfrak{B}}(Ad(g)x)}.$$

For proving that the smooth unitary representation $(\pi, \rho^\pi, \mathcal{H})$ satisfies $\rho^\pi(x)|_{\mathfrak{B}} = \rho^{\mathfrak{B}}(x)$, for every $x \in \mathfrak{g}_{\mathbb{C}}$ of degree a, apply Lemma 15 with

$$L = \mathfrak{B}, \;\; P_1 = i\overline{\alpha(a)}^{\frac{1}{2}}\rho^\pi(x)|_{\mathcal{H}^\infty} \;\text{ and }\; P_2 = i\overline{\alpha(a)}^{\frac{1}{2}}\overline{\rho^{\mathfrak{B}}(x)}$$

. This implies that $\rho^\pi(x)$ is unique.

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