Lie Symmetry Analysis of the Charney-Hasegawa-Mima equation

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Content

Geophysical background:

- Charney J. G., *On the Scale of Atmospheric Motions*, Astrophysical Institute, University of Oslo (1948).
- LaCasce J. H., *Atmosphere-Ocean Dynamics*, Dept. of Geosciences, University of Oslo (2020).
- Pedlosky J., *Geophysical Fluid Dynamics*, Springer (1987).

Methodology:

- Olver, P., *Equivalence, Invariants and Symmetry*, Cambridge University Press (1995).

Lie Symmetries of the CHM equation for the special case $\beta = F = 1$:

- Hounkonnou M. N., Kabir M. M., *Hasegawa - Mima - Charney - Obukhov Equation: Symmetry Reductions and Solutions*, Int. J. Contemp. Math. Sciences, Vol.3, p. 145 - 157 (2008).

The Charney-Hasegawa-Mima equation:

$$\frac{\partial}{\partial t}(\Delta u - Fu) + \beta \frac{\partial u}{\partial x} + [u, \Delta u] = 0,$$

$u = u(t, x, y)$	$\dots$	stream function
t	$\dots$	temporal coordinate
(x, y)	$\dots$	spatial coordinates
$\beta > 0,\ F \geq 0$	$\dots$	constants

$$\beta = \beta_0 \frac{L^2}{U}, \quad \beta_0 = \frac{2\Omega \cos\theta}{R_E}, \quad F = \left(\frac{L}{R}\right)^2,$$

$$R = \frac{\sqrt{gD}}{f}$$ – Rossby radius of deformation

Ω – Earth's rotation rate, θ – latitude, R_E – Earth's radius

$D/L/U$ – scales $\sim (10km/1000km/10m \cdot s^{-1})$

The Coriolis parameter $f = 2\Omega \sin\theta$ - influence of the Coriolis force on the fluid. $f(\theta) \approx f(\theta_0) + \beta_0 y$, where $y \equiv R_E(\theta - \theta_0)$.

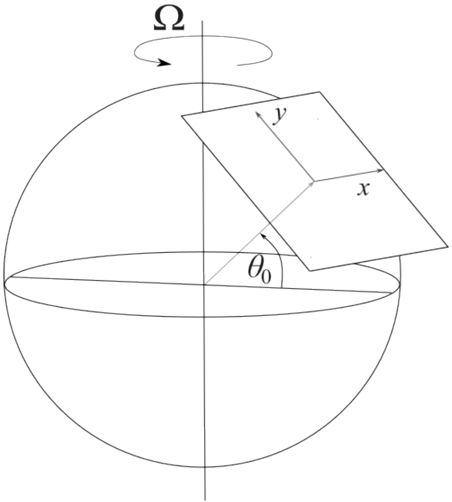

Figure: β - plane model of the CHM equation

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- Hydrostatic balance between pressure gradient and gravity

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- Geostrophic balance between horizontal pressure gradient and Coriolis pressure.
- Small dimensionless Rossby and temporal Rossby numbers

$$R_o = \frac{U}{fL}, \quad R_T = \frac{1}{fT}, \quad U/T - \text{horizontal velocity/time scale.}$$

Infinitesimal method

General n-th order system of PDEs:

$$\Delta_\kappa(\mathbf{x}, \mathbf{u}^{(n)}) = 0, \quad \kappa = 1, \ldots, m,$$

where $(\mathbf{x}, \mathbf{u}^{(n)}) \in J^n Y$.
Identification with variety

$$\mathcal{S}_\Delta = \{(\mathbf{x}, \mathbf{u}^{(n)}) \mid \Delta_\kappa(\mathbf{x}, \mathbf{u}^{(n)}) = 0, \quad \kappa = 1, \ldots, m, \}$$

- Solution - any function $s(\mathbf{x})$, such that its graph of n-th prolongation lies in $\mathcal{S}_\Delta$ (i.e. $(\mathbf{x}, s^{(n)}(\mathbf{x})) \subseteq \mathcal{S}_\Delta$).
- The Lie point symmetry - any smooth point transformation (action of the Lie group G), which maps smooth solutions to smooth solutions.

$$\implies \mathcal{S}_\Delta \text{ is } G-\text{invariant.}$$

Infinitesimal method

Lie algebra $\mathfrak{g}$ associated to Lie group G is spanned by infinitesimal generators

$$X = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi^\alpha(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial u^\alpha}. \quad (1)$$

Then n-th prolongation of the vector field X has a form

$$X^{(n)} = \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_{\#J=j=0}^{n} \phi_J^\alpha(\mathbf{x}, \mathbf{u}^{(j)}) \frac{\partial}{\partial u_J^\alpha}, \quad (2)$$

with coefficients

$$\phi_J^\alpha = D_J Q^\alpha + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha, \quad \alpha = 1, \ldots, q, \quad (3)$$

where $D_J Q^\alpha$ is the total derivative of the characteristic function

$$Q^\alpha = \phi^\alpha(\mathbf{x}, \mathbf{u}) - \sum_{i=1}^{p} \xi^i(\mathbf{x}, \mathbf{u}) \frac{\partial u^\alpha}{\partial x^i}, \quad \alpha = 1, \ldots, q. \quad (4)$$

Infinitesimal symmetry criterion

Theorem

A connected Lie group G is a symmetry group of the fully regular system of differential equations $\Delta = 0$, if and only if the classical infinitesimal symmetry conditions

$$X^{(n)}(\Delta_\kappa) = 0, \quad \kappa = 1, \ldots, r, \quad \textit{whenever} \;\; \Delta = 0,$$

hold for every infinitesimal generator $X \in \mathfrak{g}$ of G.

Algorithm

- Set symmetry criterion for given PDE or system of PDEs

$$X^{(n)}(\Delta_\kappa) = 0, \quad \kappa = 1, \ldots, r, \quad \text{whenever} \;\; \Delta = 0.$$

- Derive coefficients ϕ_J^α.
- Solve the system of *determining equations*.

The CHM equation with applied Laplacian and Jacobian:

$$u_{txx} + u_{tyy} - Fu_t + \beta u_x + u_x u_{xxy} + u_x u_{yyy} - u_y u_{xxx} - u_y u_{xyy} = 0.$$

Infinitesimal generator $X(t, x, y, u)$ over $J^0 Y$

$$X(t, x, y, u) = \tau \frac{\partial}{\partial t} + \chi \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial u}.$$

The third prolongation $X^{(3)}(t, x, y, u)$ over $J^3 Y$

$$X^{(3)} = \tau \frac{\partial}{\partial t} + \chi \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial u} + \mu^t \frac{\partial}{\partial u_t} + \mu^x \frac{\partial}{\partial u_x} + \mu^y \frac{\partial}{\partial u_y} +$$

$$+ \mu^{tt} \frac{\partial}{\partial u_{tt}} + \mu^{tx} \frac{\partial}{\partial u_{tx}} + \ldots + \mu^{xyy} \frac{\partial}{\partial u_{xyy}} + \mu^{yyy} \frac{\partial}{\partial u_{yyy}}.$$

Infinitesimal symmetry criterion:

$$X^{(3)}(\Delta_1) = \mu^{txx} + \mu^{tyy} - F\mu^t + \beta\mu^x + \mu^x(u_{xxy} + u_{yyy}) +$$

$$+ u_x(\mu^{xxy} + \mu^{yyy}) - \mu^y(u_{xxx} + u_{xyy}) - u_y(\mu^{xxx} + \mu^{xyy}) = 0, \quad (5)$$

whenever

$$u_{txx} + u_{tyy} + Fu_t - \beta u_x - u_x u_{xxy} - u_x u_{yyy} + u_y u_{xxx} + u_y u_{xyy} = 0. \quad (6)$$

We derive the coefficients $\mu^t, \mu^x, \ldots, \mu^{yyy}$, for example

$$\mu^t = D_t\mu - (D_t\tau)u_t - (D_t\chi)u_x - (D_t\psi)u_y,$$

$$\mu^{xxy} = D_{xxy}\mu - D_{xxy}(\tau u_t) - D_{xxy}(\chi u_x) - D_{xxy}(\psi u_y) +$$

$$+\tau u_{txxy} + \chi u_{xxxy} + \psi u_{xxyy},$$

a insert them into equation (5), together with expressed term from eq. (6)

$$u_{txx} = -u_{tyy} + Fu_t - \beta u_x - u_x u_{xxy} - u_x u_{yyy} + u_y u_{xxx} + u_y u_{xyy}.$$

We obtain the polynomial with indeterminates $u, u_t, u_x, \ldots, u_{yyy}$:

$$1 \cdot (\mu_{txx} + \mu_{tyy} - F\mu_t + \beta\mu_x) +$$

$$u_t \cdot (\mu_{xxu} + \mu_{yyu} - \beta\tau_x - 2F\chi_x) +$$

$$u_x \cdot (2\mu_{txu} + \mu_{xxy} + \mu_{yyy} - \chi_{txx} - \chi_{tyy} + F\chi_t + \beta\chi_x + \beta\tau_t) +$$

$$u_y \cdot (2\mu_{tyu} - \mu_{xxx} - \mu_{xyy} - \psi_{txx} - \psi_{tyy} + F\psi_t - \beta\psi_x) +$$

$$u_t u_x \cdot (2\mu_{xuu}) +$$

$$+$$

$$\vdots$$

$$+$$

$$u_y u_{xyy} \cdot (3\psi_y - \tau_t - \chi_x - \mu_u) +$$

$$u_y u_{yyy} \cdot (-\psi_x + \psi_x) = 0.$$

All brackets has to be equal to zero $\implies$ linear PDEs with constant coef.

Solution of linear PDEs with constant coefficients:

- `Macaulay2` – Linux software for computations in commutative algebra and algebraic geometry.
- We solve in `Macaulay2` those equations without general coefficients β, F.
- Remaining equations with general coefficients β, F can be solved easily by hand.

Algorithm core in `Macaulay2`:

- Linear partial differential equations with constant coefficients have a same structure as vectors of polynomials

$$2\chi_{xy} + 3\chi_{xx} + \psi_{yy} - 2\mu_{yu} \mapsto \begin{pmatrix} 0 \\ 2\partial_x\partial_y + 3(\partial_x)^2 \\ (\partial_y)^2 \\ -2\partial_y\partial_u \end{pmatrix},$$

- *Fundamental Principle of Ehrenpreis–Palamodov.*

System of linear PDEs has different solution depending on the F value

$$F = \left(\frac{L}{R}\right)^2 = \frac{f^2 L^2}{gD},$$

where L – characteristic length and R – *Rossby radius of deformation*.

1. $F \approx 0$ (near equator) – Rossby radius of deformation is large. Planet's rotation has dominant effect on the flow developent – zonal flows, jet streams, ocean currents.
2. $F > 0$ (mid-latitudes) – Influences of the planet's rotation and changes in the flow (caused by pressure/temperature gradient, topografy etc.) are in balance.
3. $F \gg 0$ (near north/south poles) – Rossby radius of deformation is small, planet's rotation has neglectable effect on the fluid flow.

Solution with assumption $F > 0$ (balance, mid-latitudes):

$$\tau(t) = c_1 + c_5 t,$$

$$\chi(t,x,y) = c_2 - \frac{\beta}{F} c_5 t - c_6 y,$$

$$\psi(t,x,y) = c_3 + \frac{\beta}{F} c_6 t + c_6 x,$$

$$\mu(t,x,y,u) = c_4 + \frac{\beta^2}{F^2} c_6 t + \frac{\beta}{F} c_6 x + \frac{\beta}{F} c_5 y - c_5 u.$$

Infinitesimal generators of Lie algebra $\mathfrak{g}_{sym}$:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial u},$$

$$X_5 = Ft\frac{\partial}{\partial t} - \beta t \frac{\partial}{\partial x} + (\beta y - Fu)\frac{\partial}{\partial u},$$

$$X_6 = -F^2 y \frac{\partial}{\partial x} + (\beta F t + F^2 x)\frac{\partial}{\partial y} + (\beta^2 t + \beta F x)\frac{\partial}{\partial u}.$$

Corresponding Lie point transformations (α_i – group parameter):

- Generators X_1, X_2, X_3, X_4:

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t + \alpha_1, x + \alpha_2, y + \alpha_3, u + \alpha_4), \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}.$$

- Generator X_5

$$\begin{aligned} \tilde{t} &= te^{F\alpha_5}, \\ \tilde{x} &= x + \frac{\beta t}{F}(1 - e^{F\alpha_5}), \\ \tilde{y} &= y, \\ \tilde{u} &= \frac{\beta y}{F}(1 - e^{-F\alpha_5}) + ue^{-F\alpha_5}. \end{aligned}$$

- Generator X_6, where $K = \dfrac{\beta}{F}$

$$\begin{aligned}
\tilde{t} &= t, \\
\tilde{x} &= x\cos(F^2\alpha_6) - y\sin(F^2\alpha_6) + Kt(\cos(F^2\alpha_6) - 1), \\
\tilde{y} &= x\sin(F^2\alpha_6) + y\cos(F^2\alpha_6) + Kt\sin(F^2\alpha_6), \\
\tilde{u} &= Kx\sin(F^2\alpha_6) + Ky\left(\cos(F^2\alpha_6) - 1\right) + K^2t\sin(F^2\alpha_6) + u.
\end{aligned}$$

Abbreviating F^2 and reorganizing terms

$$\begin{aligned}
\tilde{t} &= t, \\
\tilde{x} &= (x + Kt)\cos(\alpha_6) - y\sin(\alpha_6) - Kt, \\
\tilde{y} &= (x + Kt)\sin(\alpha_6) + y\cos(\alpha_6), \\
\tilde{u} &= K(x + Kt)\sin(\alpha_6) + Ky\left(\cos(\alpha_6) - 1\right) + u.
\end{aligned}$$

Geometrical interpretation

- Generators X_1, X_2, X_3, X_4 – translation in coordinates t, x, y, u.

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t + \alpha_1, x + \alpha_2, y + \alpha_3, u + \alpha_4), \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}.$$

– doesn't have impact on the fluid velocity fields

- Generator X_5 – time-length contraction/extension in (t, x) plane

$$\tilde{t} = te^{F\alpha_5}, \;\; \tilde{x} = x + \frac{\beta t}{F}(1 - e^{F\alpha_5})$$

$$\implies \quad \tilde{t}_2 - \tilde{t}_1 = (t_2 - t_1)e^{F\alpha_5}$$

1. $\alpha_5 > 0$ – time extension
2. $\alpha_5 < 0$ – time contraction

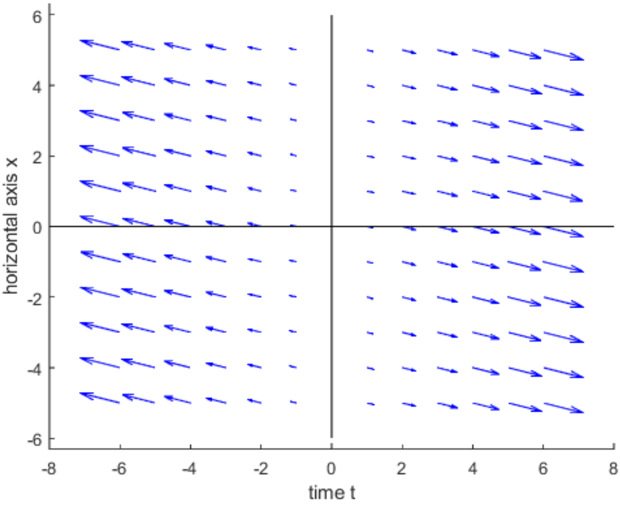

Figure: Vector field X_5, $\beta = 0.25$, $F = 1$

Length/time change ratio: - phase velocity $K = \frac{\beta}{F}$ is preserved (x dir.)

$$\frac{\tilde{x} - x}{\tilde{t} - t} = -\frac{\beta}{F}, \qquad \tilde{x} = x - \frac{\beta}{F}(\tilde{t} - t).$$

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1) Example of stream function

$$u = y \implies \tilde{u} = \tilde{y}\left(e^{-F\alpha_6} + \frac{\beta}{F}\left(1 - e^{-F\alpha_6}\right)\right)$$

Change in flow velocity fields:

$$\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = (1, 0) \implies \left(\frac{\partial \tilde{u}}{\partial y}, -\frac{\partial \tilde{u}}{\partial x}\right) = \left(e^{-F\alpha_6} + \frac{\beta}{F}\left(1 - e^{-F\alpha_6}\right), 0\right)$$

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$$\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = (1, 0) \implies \left(\frac{\partial \tilde{u}}{\partial y}, -\frac{\partial \tilde{u}}{\partial x}\right) = \left(e^{-F\alpha_6} + \frac{\beta}{F}\left(1 - e^{-F\alpha_6}\right), 0\right)$$

2) Example of stream function

$$u = x \implies \tilde{u} = \frac{\beta}{F}\tilde{y}(1 - e^{-F\alpha_6}) + \tilde{x}e^{-F\alpha_6}$$

Change in flow velocity fields:

$$\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = (0, -1) \implies \left(\frac{\partial \tilde{u}}{\partial y}, -\frac{\partial \tilde{u}}{\partial x}\right) = \left(\frac{\beta}{F}(1 - e^{-F\alpha_6}), e^{-F\alpha_6}\right)$$

- Generator X_6 – rotation around the origin with respect to the phase velocity in x direction $K = \frac{\beta}{F}$.

$$\begin{aligned}
\tilde{t} &= t, \\
\tilde{x} &= (x + Kt)\cos(\alpha_6) - y\sin(\alpha_6) - Kt, \\
\tilde{y} &= (x + Kt)\sin(\alpha_6) + y\cos(\alpha_6), \\
\tilde{u} &= K(x + Kt)\sin(\alpha_6) + Ky\,(\cos(\alpha_6) - 1) + u.
\end{aligned}$$

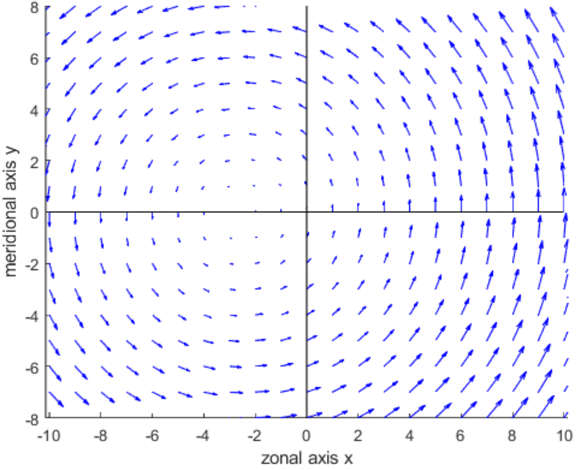


Figure: Vector field X_6, $\beta = 0.25$, $F = 1$, $t = 10$

Solution with assumption $F \gg 0$ (neglectable rotation influence):

$$\tau(t) = c_1 + c_5 t,$$

$$\chi(y) = c_2 - c_6 y,$$

$$\psi(x) = c_3 + c_6 x,$$

$$\mu(u) = c_4 - c_5 u.$$

Infinitesimal generators of Lie algebra $\mathfrak{g}_{sym}$:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial u},$$

$$X_5 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u},$$

$$X_6 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Corresponding Lie-point transformations and interpretation

- Generators X_1, X_2, X_3, X_4 – translations
- Generator X_5 – scaling in temporary coordinate t

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = \left(te^{\alpha_5}, x, y, ue^{-\alpha_5}\right).$$

- Generator X_6 – rotation in β-plane

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t, x\cos(\alpha_6), y\sin(\alpha_6), u).$$

Solution with assumption $F \approx 0$ (dominant rotation influence):

$$\tau(t) = c_1,$$

$$\chi(t,x,y) = -h(t),$$

$$\psi(t,x,y) = c_2,$$

$$\mu(t,x,y,u) = f(t) + h'(t)y.$$

Infinitesimal generators of Lie algebra $\mathfrak{g}_{sym}$:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = -h(t)\frac{\partial}{\partial x} + h'(t)y\frac{\partial}{\partial u}, \quad X_4 = f(t)\frac{\partial}{\partial u}.$$

Corresponding Lie-point transformations and interpretation

- Generators X_1, X_2 – translations in t, y coordinates
- Generator X_3 – time-dependent motion along x-coordinate – by setting $\alpha_3 = t,\ h(t)$ represents planet's pheripheral velocity (zonal direction)

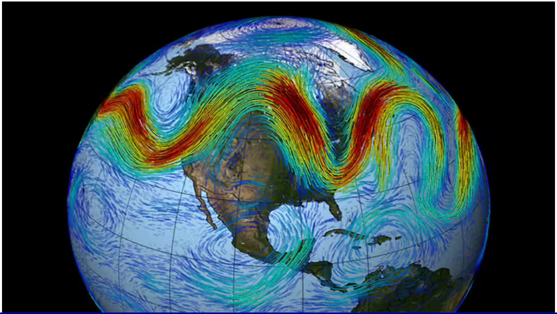
$$\begin{aligned}
\tilde{t} &= t, \\
\tilde{x} &= -h(t)\alpha_3 + x, \\
\tilde{y} &= y, \\
\tilde{u} &= h'(t)y\alpha_3 + u.
\end{aligned}$$

- Generator X_4 – any time-dependent change in stream function values

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t, x, y, f(t)\alpha_4 + u).$$

Conclusion

- For reasonably large $F > 0$ (mid-latitudes), the CHM equation has Lie symmetries of translations, time-scaling and rotation, which preserve phase velocity $\frac{\beta}{F}$ in x-direction.
- Special cases of F value (rather theoretical meaning):
  1. $F \approx 0$ (planet's rotation is dominant, equator) $\implies$ time-dependent motion symmetries along x and u coordinates
  2. $F \gg 0$ (planet's rotation is neglectable, poles) $\implies$ rotation symmetry in spatial (x, y) coordinates and scaling in temporary t coordinate



Conclusion

- Translation symmetry is present in all F instances.
- The most general solution of the CHM equation (for particular F instance) – application of all G_{sym} transformations to coordinates of the known solution $u = u(t, x, y)$.

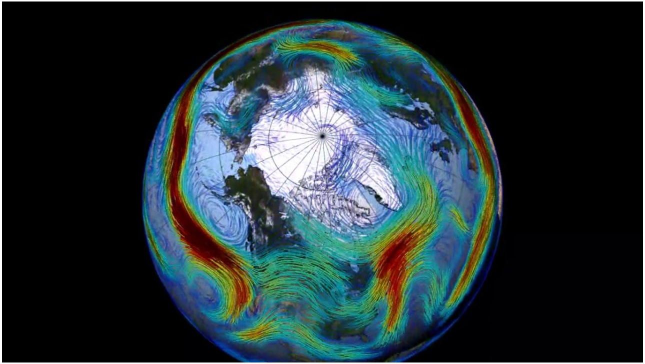


Figure: Rossby waves