Lie Symmetry Analysis of the Charney-Hasegawa-Mima equation

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July 6, 2023

Content

Geophysical background:

- Charney J. G., *On the Scale of Atmospheric Motions*, Astrophysical Institute, University of Oslo (1948).
- LaCasce J. H., *Atmosphere-Ocean Dynamics*, Dept. of Geosciences, University of Oslo (2020).
- Pedlosky J., *Geophysical Fluid Dynamics*, Springer (1987).

Methodology:

- Olver, P., *Equivalence, Invariants and Symmetry*, Cambridge University Press (1995).

Lie Symmetries of the CHM equation for the special case $\beta = F = 1$:

- Hounkonnou M. N., Kabir M. M., *Hasegawa - Mima - Charney - Obukhov Equation: Symmetry Reductions and Solutions*, Int. J. Contemp. Math. Sciences, Vol.3, p. 145 - 157 (2008).

The Charney-Hasegawa-Mima equation:

$$\frac{\partial}{\partial t}(\quad u \quad Fu) + \beta \frac{\partial u}{\partial x} + [u, \quad u] = 0,$$

$u = u(t, x, y)$ ... stream function
t ... temporal coordinate
(x, y) ... spatial coordinates
$\beta > 0, \ F \quad 0$ ... constants

$$\beta = \beta_0 \frac{L^2}{U}, \quad \beta_0 = \frac{2 \quad \cos\theta}{R_E}, \quad F = \left(\frac{L}{R}\right)^2,$$

$$R = \frac{\sqrt{gD}}{f} \quad \text{Rossby radius of deformation}$$

Earth's rotation rate, θ latitude, R_E Earth's radius

$D/L/U$ scales $(10km/1000km/10m \ \ s \ \ ^{1})$

The Coriolis parameter $f = 2 \quad \sin\theta$ - influence of the Coriolis force on the fluid. $f(\theta) \quad f(\theta_0) + \beta_0 y$, where $y \quad R_E(\theta \quad \theta_0)$.

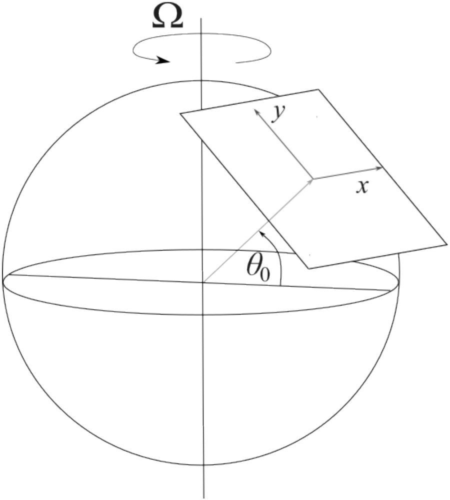

Figure: β - plane model of the CHM equation

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- Hydrostatic balance between pressure gradient and gravity

$$\frac{\partial p}{\partial z} = \quad \rho g.$$

- Geostrophic balance between horizontal pressure gradient and Coriolis pressure.
- Small dimensionless Rossby and temporal Rossby numbers

$$R_o = \frac{U}{fL}, \quad R_T = \frac{1}{fT}, \quad U/T \quad \text{horizontal velocity/time scale.}$$

Infinitesimal method

General n-th order system of PDEs:

$$(\mathbf{x}, \mathbf{u}^{(n)}) = 0, \quad \kappa = 1, \ldots, m,$$

where $(\mathbf{x}, \mathbf{u}^{(n)})$ 2 $J^n Y$.
Identification with variety

$$S \quad = f(\mathbf{x}, \mathbf{u}^{(n)})\ j \qquad (\mathbf{x}, \mathbf{u}^{(n)}) = 0, \quad \kappa = 1, \ldots, m, g$$

- Solution - any function $s(\mathbf{x})$, such that its graph of n-th prolongation lies in S (i.e. $(\mathbf{x}, s^{(n)}(\mathbf{x}))$ S).
- The Lie point symmetry - any smooth point transformation (action of the Lie group G), which maps smooth solutions to smooth solutions.

=) S is G invariant.

Infinitesimal method

Lie algebra $\mathfrak{g}$ associated to Lie group G is spanned by infinitesimal generators

$$X = \sum_{i=1}^{p} \xi^{i}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^{i}} + \sum_{=1}^{q} \phi \; (\mathbf{x},\mathbf{u})\frac{\partial}{\partial u}. \qquad (1)$$

Then n-th prolongation of the vector field X has a form

$$X^{(n)} = \sum_{i=1}^{p} \xi^{i}(\mathbf{x},\mathbf{u})\frac{\partial}{\partial x^{i}} + \sum_{=1}^{q} \sum_{\#J=j=0}^{n} \phi_{J}(\mathbf{x},\mathbf{u}^{(j)})\frac{\partial}{\partial u_{J}}, \qquad (2)$$

with coefficients

$$\phi_{J} = D_{J}Q \;\; + \sum_{i=1}^{p} \xi^{i} u_{J;i}, \quad \alpha = 1,\ldots,q, \qquad (3)$$

where $D_J Q$ is the total derivative of the characteristic function

$$Q \;\; = \phi \; (\mathbf{x},\mathbf{u}) \quad \sum_{i=1}^{p} \xi^{i}(\mathbf{x},\mathbf{u})\frac{\partial u}{\partial x^{i}}, \quad \alpha = 1,\ldots,q. \qquad (4)$$

Infinitesimal symmetry criterion

Theorem

A connected Lie group G is a symmetry group of the fully regular system of di erential equations = 0, if and only if the classical in nitesimal symmetry conditions

$$X^{(n)}(\quad) = 0, \quad \kappa = 1, \ldots, r, \quad \textit{whenever} \quad = 0,$$

hold for every in nitesimal generator X 2 g of G.

Algorithm

- Set symmetry criterion for given PDE or system of PDEs

$$X^{(n)}(\quad) = 0, \quad \kappa = 1, \ldots, r, \quad \text{whenever} \quad = 0.$$

- Derive coefficients ϕ_J.
- Solve the system of *determining equations*.

The CHM equation with applied Laplacian and Jacobian:

$$u_{txx} + u_{tyy} \quad F u_t + \beta u_x + u_x u_{xxy} + u_x u_{yyy} \quad u_y u_{xxx} \quad u_y u_{xyy} = 0.$$

Infinitesimal generator $X(t,x,y,u)$ over $J^0 Y$

$$X(t,x,y,u) = \tau \frac{\partial}{\partial t} + \chi \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial u}.$$

The third prolongation $X^{(3)}(t,x,y,u)$ over $J^3 Y$

$$X^{(3)} = \tau \frac{\partial}{\partial t} + \chi \frac{\partial}{\partial x} + \psi \frac{\partial}{\partial y} + \mu \frac{\partial}{\partial u} + \mu^{t} \frac{\partial}{\partial u_t} + \mu^{x} \frac{\partial}{\partial u_x} + \mu^{y} \frac{\partial}{\partial u_y} +$$

$$+ \mu^{tt} \frac{\partial}{\partial u_{tt}} + \mu^{tx} \frac{\partial}{\partial u_{tx}} + \ldots + \mu^{xyy} \frac{\partial}{\partial u_{xyy}} + \mu^{yyy} \frac{\partial}{\partial u_{yyy}}.$$

Infinitesimal symmetry criterion:

$$X^{(3)}(\quad_1) = \mu^{txx} + \mu^{tyy} \quad F\mu^{t} + \beta\mu^{x} + \mu^{x}(u_{xxy} + u_{yyy}) +$$

$$+ u_x(\mu^{xxy} + \mu^{yyy}) \quad \mu^{y}(u_{xxx} + u_{xyy}) \quad u_y(\mu^{xxx} + \mu^{xyy}) = 0, \qquad (5)$$

whenever

$$u_{txx} + u_{tyy} + Fu_t \quad \beta u_x \quad u_x u_{xxy} \quad u_x u_{yyy} + u_y u_{xxx} + u_y u_{xyy} = 0. \qquad (6)$$

We derive the coefficients $\mu^{t}, \mu^{x}, \ldots, \mu^{yyy}$, for example

$$\mu^{t} = D_t\mu \quad (D_t\tau)u_t \quad (D_t\chi)u_x \quad (D_t\psi)u_y,$$

$$\mu^{xxy} = D_{xxy}\mu \quad D_{xxy}(\tau u_t) \quad D_{xxy}(\chi u_x) \quad D_{xxy}(\psi u_y) +$$

$$+\tau u_{txxy} + \chi u_{xxxy} + \psi u_{xxyy},$$

a insert them into equation (5), together with expressed term from eq. (6)

$$u_{txx} = \quad u_{tyy} + Fu_t \quad \beta u_x \quad u_x u_{xxy} \quad u_x u_{yyy} + u_y u_{xxx} + u_y u_{xyy}.$$

We obtain the polynomial with indeterminates $u, u_t, u_x, \ldots, u_{yyy}$:

$$1 \quad (\mu_{txx} + \mu_{tyy} \quad F\mu_t + \beta\mu_x)+$$

$$u_t \quad (\mu_{xxu} + \mu_{yyu} \quad \beta\tau_x \quad 2F\chi_x)+$$

$$u_x \quad (2\mu_{txu} + \mu_{xxy} + \mu_{yyy} \quad \chi_{txx} \quad \chi_{tyy} + F\chi_t + \beta\chi_x + \beta\tau_t)+$$

$$u_y \quad (2\mu_{tyu} \quad \mu_{xxx} \quad \mu_{xyy} \quad \psi_{txx} \quad \psi_{tyy} + F\psi_t \quad \beta\psi_x)+$$

$$u_t u_x \quad (2\mu_{xuu})+$$

$$+$$

$$\vdots$$

$$+$$

$$u_y u_{xyy} \quad (3\psi_y \quad \tau_t \quad \chi_x \quad \mu_u)+$$

$$u_y u_{yyy} \quad (\quad \psi_x + \psi_x) = 0.$$

All brackets has to be equal to zero =) linear PDEs with constant coef.

Solution of linear PDEs with constant coefficients:

- `Macaulay2` – Linux software for computations in commutative algebra and algebraic geometry.
- We solve in `Macaulay2` those equations without general coefficients β, F.
- Remaining equations with general coefficients β, F can be solved easily by hand.

Algorithm core in `Macaulay2`:

- Linear partial differential equations with constant coefficients have a same structure as vectors of polynomials

$$2\chi_{xy} + 3\chi_{xx} + \psi_{yy} \quad 2\mu_{yu} \mapsto \begin{pmatrix} 0 \\ 2\partial_x\partial_y + 3(\partial_x)^2 \\ (\partial_y)^2 \\ 2\partial_y\partial_u \end{pmatrix},$$

- *Fundamental Principle of Ehrenpreis{Palamodov.*

System of linear PDEs has different solution depending on the F value

$$F = \left(\frac{L}{R}\right)^2 = \frac{f^2L^2}{gD},$$

where L – characteristic length and R – *Rossby radius of deformation*.

1. F 0 (near equator) – Rossby radius of deformation is large. Planet's rotation has dominant effect on the flow developent – zonal flows, jet streams, ocean currents.
2. $F > 0$ (mid-latitudes) – Influences of the planet's rotation and changes in the flow (caused by pressure/temperature gradient, topografy etc.) are in balance.
3. F 0 (near north/south poles) – Rossby radius of deformation is small, planet's rotation has neglectable effect on the fluid flow.

Solution with assumption $F > 0$ (balance, mid-latitudes):

$$\tau(t) = c_1 + c_5 t,$$

$$\chi(t,x,y) = c_2 - \frac{\beta}{F} c_5 t - c_6 y,$$

$$\psi(t,x,y) = c_3 + \frac{\beta}{F} c_6 t + c_6 x,$$

$$\mu(t,x,y,u) = c_4 + \frac{\beta^2}{F^2} c_6 t + \frac{\beta}{F} c_6 x + \frac{\beta}{F} c_5 y - c_5 u.$$

Infinitesimal generators of Lie algebra $\mathfrak{g}_{sym}$:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial u},$$

$$X_5 = Ft\frac{\partial}{\partial t} - \beta t \frac{\partial}{\partial x} + (\beta y - Fu)\frac{\partial}{\partial u},$$

$$X_6 = -F^2 y \frac{\partial}{\partial x} + (\beta F t + F^2 x)\frac{\partial}{\partial y} + (\beta^2 t + \beta F x)\frac{\partial}{\partial u}.$$

Corresponding Lie point transformations (α_i – group parameter):

- Generators X_1, X_2, X_3, X_4:

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t + \alpha_1, x + \alpha_2, y + \alpha_3, u + \alpha_4), \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \; 2 \; \mathrm{R}.$$

- Generator X_5

$$\begin{aligned}
\tilde{t} &= te^{F \quad 5}, \\
\tilde{x} &= x + \frac{\beta t}{F}(1 \quad e^{F \quad 5}), \\
\tilde{y} &= y, \\
\tilde{u} &= \frac{\beta y}{F}(1 \quad e^{\quad F \quad 5}) + ue^{\quad F \quad 5}.
\end{aligned}$$

- Generator X_6, where $K = \dfrac{\beta}{F}$

$$\begin{aligned}
\tilde{t} &= t, \\
\tilde{x} &= x\cos(F^2\alpha_6) \quad y\sin(F^2\alpha_6) + Kt(\cos(F^2\alpha_6) \quad 1), \\
\tilde{y} &= x\sin(F^2\alpha_6) + y\cos(F^2\alpha_6) + Kt\sin(F^2\alpha_6), \\
\tilde{u} &= Kx\sin(F^2\alpha_6) + Ky\left(\cos(F^2\alpha_6) \quad 1\right) + K^2 t\sin(F^2\alpha_6) + u.
\end{aligned}$$

Abbreviating F^2 and reorganizing terms

$$\begin{aligned}
\tilde{t} &= t, \\
\tilde{x} &= (x + Kt)\cos(\alpha_6) \quad y\sin(\alpha_6) \quad Kt, \\
\tilde{y} &= (x + Kt)\sin(\alpha_6) + y\cos(\alpha_6), \\
\tilde{u} &= K(x + Kt)\sin(\alpha_6) + Ky\,(\cos(\alpha_6) \quad 1) + u.
\end{aligned}$$

Geometrical interpretation

- Generators X_1, X_2, X_3, X_4 – translation in coordinates t, x, y, u.

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t + \alpha_1, x + \alpha_2, y + \alpha_3, u + \alpha_4), \quad \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}.$$

– doesn't have impact on the fluid velocity fields

- Generator X_5 – time-length contraction/extension in (t, x) plane

$$\tilde{t} = t e^{F\alpha_5}, \;\; \tilde{x} = x + \frac{\beta t}{F}(1 - e^{F\alpha_5})$$

$$\Longrightarrow \quad \tilde{t}_2 - \tilde{t}_1 = (t_2 - t_1) e^{F\alpha_5}$$

1. $\alpha_5 > 0$ – time extension
2. $\alpha_5 < 0$ – time contraction

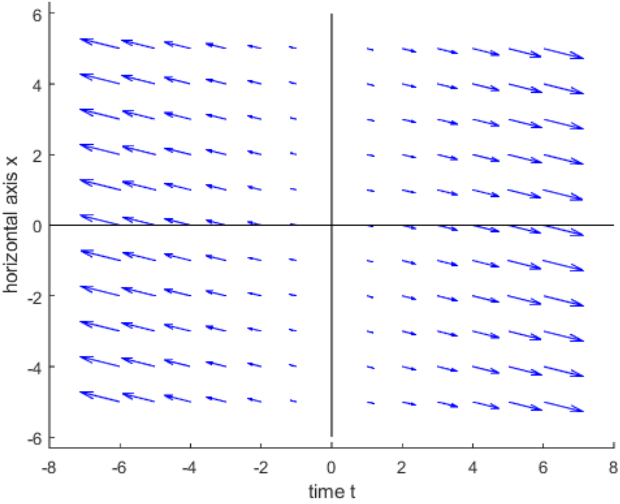

Figure: Vector field X_5, $\beta = 0.25$, $F = 1$

Length/time change ratio: - phase velocity $K = \frac{\beta}{F}$ is preserved (x dir.)

$$\frac{\tilde{x} \quad x}{\tilde{t} \quad t} = \quad \frac{\beta}{F}, \qquad \tilde{x} = x \quad \frac{\beta}{F}(\tilde{t} \quad t).$$

Length/time change ratio: - phase velocity $K = \frac{\beta}{F}$ is preserved (x dir.)

$$\frac{\tilde{x} - x}{\tilde{t} - t} = -\frac{\beta}{F}, \qquad \tilde{x} = x - \frac{\beta}{F}(\tilde{t} - t).$$

1) Example of stream function

$$u = y \implies \tilde{u} = \tilde{y}\left(e^{-F\epsilon_6} + \frac{\beta}{F}\left(1 - e^{-F\epsilon_6}\right)\right)$$

Change in flow velocity fields:

$$\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = (1, 0) \implies \left(\frac{\partial \tilde{u}}{\partial y}, -\frac{\partial \tilde{u}}{\partial x}\right) = \left(e^{-F\epsilon_6} + \frac{\beta}{F}\left(1 - e^{-F\epsilon_6}\right), 0\right)$$

Length/time change ratio: - phase velocity $K = \frac{\beta}{F}$ is preserved (x dir.)

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Change in flow velocity fields:

$$\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = (1, 0) \Longrightarrow \left(\frac{\partial \tilde{u}}{\partial y}, -\frac{\partial \tilde{u}}{\partial x}\right) = \left(e^{-F\epsilon} + \frac{\beta}{F}\left(1 - e^{-F\epsilon}\right), 0\right)$$

2) Example of stream function

$$u = x \Longrightarrow \tilde{u} = \frac{\beta}{F}\tilde{y}(1 - e^{-F\epsilon}) + \tilde{x}e^{-F\epsilon}$$

Change in flow velocity fields:

$$\left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right) = (0, -1) \Longrightarrow \left(\frac{\partial \tilde{u}}{\partial y}, -\frac{\partial \tilde{u}}{\partial x}\right) = \left(\frac{\beta}{F}(1 - e^{-F\epsilon}), e^{-F\epsilon}\right)$$

- Generator X_6 – rotation around the origin with respect to the phase velocity in x direction $K = \dfrac{\beta}{F}$.

$$
\begin{aligned}
\tilde{t} &= t,\\
\tilde{x} &= (x + Kt)\cos(\alpha_6) - y\sin(\alpha_6) - Kt,\\
\tilde{y} &= (x + Kt)\sin(\alpha_6) + y\cos(\alpha_6),\\
\tilde{u} &= K(x + Kt)\sin(\alpha_6) + Ky\,(\cos(\alpha_6) - 1) + u.
\end{aligned}
$$

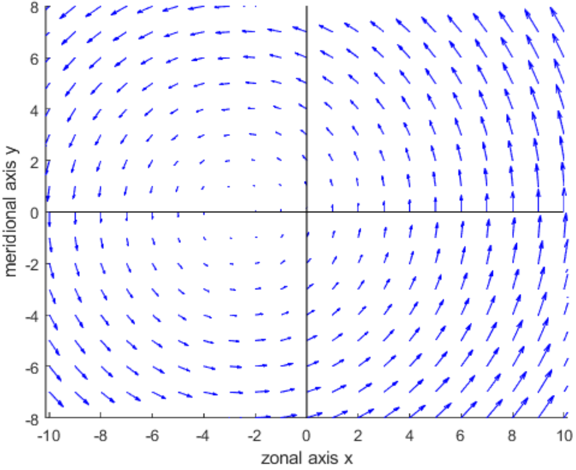


Figure: Vector field X_6, $\beta = 0.25$, $F = 1$, $t = 10$

Solution with assumption F 0 (neglectable rotation influence):

$$\tau(t) = c_1 + c_5 t,$$

$$\chi(y) = c_2 \quad c_6 y,$$

$$\psi(x) = c_3 + c_6 x,$$

$$\mu(u) = c_4 \quad c_5 u.$$

Infinitesimal generators of Lie algebra $\mathfrak{g}_{sym}$:

$$X_1 = \frac{\partial}{\partial t},\ X_2 = \frac{\partial}{\partial x},\ X_3 = \frac{\partial}{\partial y},\ X_4 = \frac{\partial}{\partial u},$$

$$X_5 = t\frac{\partial}{\partial t} \quad u\frac{\partial}{\partial u},$$

$$X_6 = \quad y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Corresponding Lie-point transformations and interpretation

- Generators X_1, X_2, X_3, X_4 – translations
- Generator X_5 – scaling in temporary coordinate t

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = \left(te^{\;5}, x, y, ue^{\quad 5}\right).$$

- Generator X_6 – rotation in β-plane

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t, x\cos(\alpha_6), y\sin(\alpha_6), u).$$

Solution with assumption F 0 (dominant rotation influence):

$$\tau(t) = c_1,$$

$$\chi(t,x,y) = \quad h(t),$$

$$\psi(t,x,y) = c_2,$$

$$\mu(t,x,y,u) = f(t) + h'(t)y.$$

Infinitesimal generators of Lie algebra $\mathfrak{g}_{sym}$:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial y},$$

$$X_3 = \quad h(t)\frac{\partial}{\partial x} + h'(t)y\frac{\partial}{\partial u}, \quad X_4 = f(t)\frac{\partial}{\partial u}.$$

Corresponding Lie-point transformations and interpretation

- Generators X_1, X_2 – translations in t, y coordinates
- Generator X_3 – time-dependent motion along x-coordinate – by setting $\alpha_3 = t,\ h(t)$ represents planet's pheripheral velocity (zonal direction)

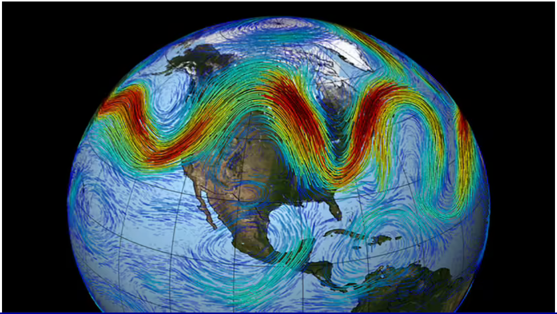
$$\begin{aligned} \tilde{t} &= t, \\ \tilde{x} &= h(t)\alpha_3 + x, \\ \tilde{y} &= y, \\ \tilde{u} &= h'(t) y \alpha_3 + u. \end{aligned}$$

- Generator X_4 – any time-dependent change in stream function values

$$(\tilde{t}, \tilde{x}, \tilde{y}, \tilde{u}) = (t, x, y, f(t)\alpha_4 + u).$$

Conclusion

- For reasonably large $F > 0$ (mid-latitudes), the CHM equation has Lie symmetries of translations, time-scaling and rotation, which preserve phase velocity $\frac{\beta}{F}$ in x-direction.
- Special cases of F value (rather theoretical meaning):
    1. F 0 (planet's rotation is dominant, equator) =) time-dependent motion symmetries along x and u coordinates
    2. F 0 (planet's rotation is neglectable, poles) =) rotation symmetry in spatial (x, y) coordinates and scaling in temporary t coordinate



Conclusion

- Translation symmetry is present in all F instances.
- The most general solution of the CHM equation (for particular F instance) – application of all G_{sym} transformations to coordinates of the known solution $u = u(t, x, y)$.

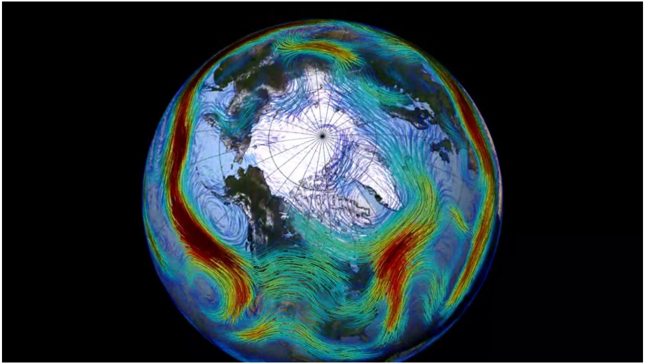


Figure: Rossby waves