Integrable quantum field theories between modular theory and the Yang–Baxter equation

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partly joint work with Ricardo Correa da Silva

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Geometric Methods in Physics, Białowieża

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- Let ϕ be a quantum field on spacetime M with vacuum Ω, and $\mathcal{O} \subset M$ open (localization region)

$$H_{\mathcal{O}} := \{\phi(f)\Omega : f \in C^\infty_{c,\mathbb{R}}(M), \operatorname{supp}(f) \subset \mathcal{O}\}^-$$

Standard subspaces and modular theory

- Given a standard subspace, have Tomita operator

$$S_H : H + iH \to H + iH, \qquad S_H(h_1 + ih_2) = h_1 - ih_2.$$

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- Given strongly continuous unitary one-parameter group $V(t) = e^{itX}$ and antiunitary involution J with $[J, V(t)] = 0$,

$$H := \ker(1 - Je^{X}) \qquad \text{is standard.}$$

- Every standard subspace is of this form.
- In particular, may generate standard subspaces from unitary group representations (Poincaré group, deSitter group, Möbius group, ...)

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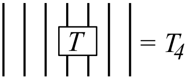
- **Notation:**

$$T_k := 1_{\mathcal{H}}^{\otimes(k-1)} \otimes T \otimes 1_{\mathcal{H}}^{\otimes(n-k-1)}$$



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$$P_{T,1} = 1, \quad P_{T,2} = 1 + T, \quad P_{T,n+1} = (1 \otimes P_{T,n})(1 + T_1 + T_1 T_2 + \ldots + T_1 \cdots T_n).$$

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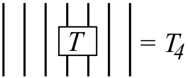

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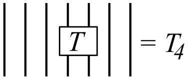

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T-twisted Fock space

$$\mathcal{F}_T(\mathcal{H}) := \bigoplus_{n \geq 0} \overline{\mathcal{H}^{\otimes n} / \ker P_{T,n}}^{\langle \cdot , \cdot \rangle_{T,n}}$$

Examples

- $T = \pm F : v \otimes w \mapsto \pm w \otimes v$ **(flip)**: $\mathcal{F}_F(\mathcal{H})$ = Bose/Fermi Fock space
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Theorem ([Jørgensen/Schmitt/Werner; Bożejko/Speicher])

Let $T = T^* \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$, $\|T\| \leq 1$.

1. *If* $\|T\| \leq \frac{1}{2}$, *then* T *is a strict twist.*
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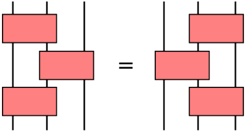

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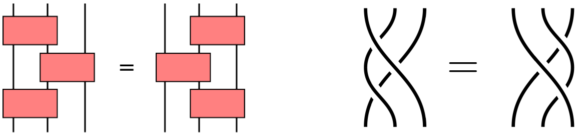

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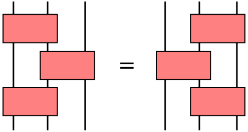

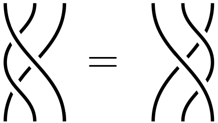

From now on: $\mathcal{H}$ Hilbert space, T arbitrary twist.

- On $\mathcal{F}_T(\mathcal{H})$, have (left) creation/annihilation operators $a_{L,T}(\xi)$, $\xi \in \mathcal{H}$:

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- $(H = H', T = 0) \to \mathcal{L}_T(H) = L\mathbb{F}_{\dim \mathcal{H}}$ [Voiculescu].
- $(H$ arbitrary, $T = 0) \to$"free Araki-Woods factor" [Shlyakhtenko]
- $(H$ arbitrary, $T = qF) \to q$-deformed Araki-Woods factors [Kumar/Skalski/Wasilewski]
- $(H = H_{\mathcal{O}}, T = F) \to \mathcal{L}_T(H) =$ free field observable algebra loc. in $\mathcal{O}$.

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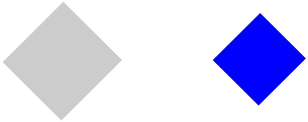

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Basic fact in QFT: For localized quantum observable algebras, need vacuum Ω **standard**, i.e. cyclic:

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- In order to have Ω separating for $\mathcal{L}_T(H)$, need **KMS-property**.

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and separating ("no annihilation operators")

$$A \in \mathcal{L}_T(H),\ A\Omega = 0 \implies A = 0.$$

For which (T, H) is Ω **standard** for $\mathcal{L}_T(H)$?

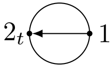
- Basic assumption in the following: Twist T **compatible** with dynamics of H:

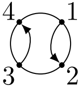
$$[T, \Delta_H^{it} \otimes \Delta_H^{it}] = 0.$$

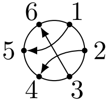
- Lemma: Ω always cyclic for $\mathcal{L}_T(H)$. But in general **not** separating.
- In order to have Ω separating for $\mathcal{L}_T(H)$, need **KMS-property**.

KMS requires analytic properties of n-point functions.

In graphical notation







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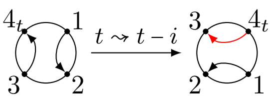

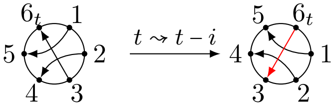

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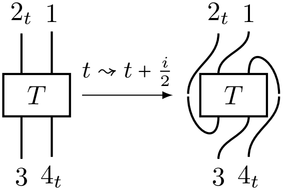

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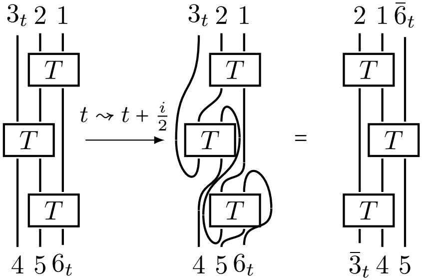

Theorem ([Correa da Silva / L 22])

$H \subset \mathcal{H}$ standard subspace, T compatible twist. Then Ω is separating for $\mathcal{L}_T(H)$ if and only if T is braided ($T_1T_2T_1 = T_2T_1T_2$) and crossing-symmetric.

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Definition

T is called **crossing-symmetric** (w.r.t. H) if for all $\psi_1, \ldots, \psi_4 \in \mathcal{H}$, the function

$$f(t) := \langle \psi_2 \otimes \psi_1,\ (\Delta_H^{it} \otimes 1) T (1 \otimes \Delta_H^{-it})(\psi_3 \otimes \psi_4) \rangle$$

has an analytic bounded continuation to the strip $\mathbb{S}_{1/2}$ and

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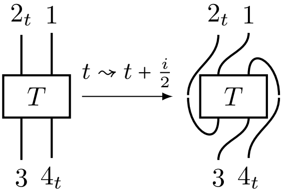

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- Crossing symmetry and Yang-Baxter equation both come from physics and are usually taken as assumptions, but can here be derived from modular theory.
- Many examples of braided crossing-symmetric twists are known. The simplest are $T = q \cdot F$ (flip), $-1 \leq q \leq 1$.
- Simplest counterexamples: $T = q \cdot 1$.

Braided twists and left-right duality

For **braided** twists ($T_1 T_2 T_1 = T_2 T_1 T_2$), there exists also a “right” version of our “left” construction $\rightsquigarrow \mathcal{R}_T(H)$.

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Proposition ([Correa da Silva / L 22])

Let T be braided and crossing symmetric.

a) *The Tomita operator S of $(\mathcal{L}_T(H), \Omega)$ is given by*

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- From our perspective, the braided and crossing-symmetric twists are the most interesting ones (classification unknown).
- Result on modular data generalizes many known results. [Eckmann/Osterwalder '73, Leyland/Roberts/Testard '78, Shlyakhtenko '97, Baumgärtel/Jurke/Lledo '02, Buchholz/L/Summers '11, L '12]
- In situation of theorem, $\mathcal{L}_T(H)$ is a factor for $\|T\| < 1$. ([Yang '23] for $\dim \mathcal{H} < \infty$, and [Correa da Silva/L '23] for $\dim \mathcal{H} = \infty$)

For braided compatible crossing-symmetric twist, $\mathcal{L}_T(H)$ satisfies *some* requirements from QFT. Now: **"upgrade":** $\mathcal{L}_T(H) \rightsquigarrow$ full QFT.

Example

Hilbert space and standard subspace:

$$\mathcal{H} = L^2(\mathbb{R}, d\theta) \otimes \mathcal{K}, \qquad \dim \mathcal{K} < \infty$$
$$\theta \longmapsto \psi^\alpha(\theta), \qquad \alpha \in \{1, \ldots, \dim \mathcal{K}\}$$
$$(\Delta_H^{it}\psi)^\alpha(\theta) = \psi^\alpha(\theta - 2\pi t),$$
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- Many crossing symmetric compatible braided twists are known (integrable systems)

$$(T_S f)(\theta_1, \theta_2) = S(\theta_1 - \theta_2) f(\theta_2, \theta_1).$$

- Matrix-valued function S has to solve YBE with spectral parameter:

$$S(\theta)_1 S(\theta + \theta')_2 S(\theta')_1 = S(\theta')_2 S(\theta + \theta')_1 S(\theta)_2.$$

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- Explicit example ("Sinh-Gordon model") $\mathcal{K} = \mathbb{C}$, $J_H z = \overline{z}$,

$$S(\theta) = \frac{\sinh\theta - ia}{\sinh\theta + ia}$$

Localisation in Rindler wedges

Representation-theoretic perspective on $\mathcal{H} = L^2(\mathbb{R}, d\theta)$ and Δ_H, J_H:

- Δ_H^{it}, J_H come from a unitary positive energy representation of the Poincaré group. Geometric meaning:

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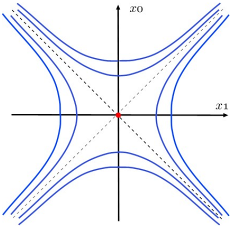

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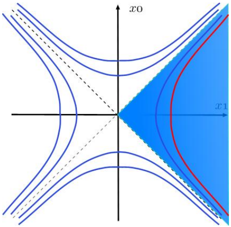
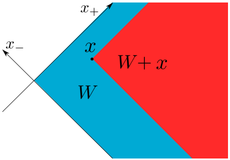


- → We should interpret $\mathcal{L}_T(H)$ as the observables localized in the Rindler wedge $W = \{(x_0, x_1) \in \mathbb{R}^2 : x_1 > |x_0|\}$.

Sharper localisation and subfactors



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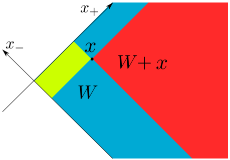

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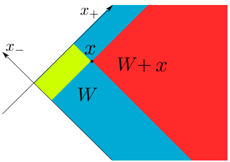

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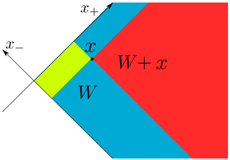

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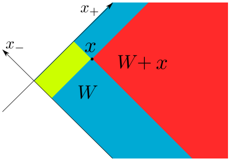

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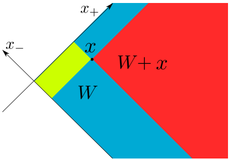

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- Need to make sure $\mathcal{A}(\mathcal{O}) \neq \mathbb{C}1$ (existence of local observables)
- Mathematical question: Given inclusion of standard subspaces $K \subset H$, analyze relative commutant of inclusion

$$\mathcal{L}_T(K) \subset \mathcal{L}_T(H).$$

This is typically a **subfactor** (both algebras have trivial centre).

Relative commutants

- Elements of $\mathcal{A}(\mathcal{O}) = \mathcal{R}_T(H')' \cap \mathcal{L}_T(K)'$ commute with both left and right fields and are very hard to compute ($\rightsquigarrow$ perturbative approaches to QFT, formfactor programme)

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Theorem (Buchholz, D'Antoni, Longo)

Suppose $A \subset B$ are von Neumann factors with Ω cyclic/separating for both, and

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"Non-locality result" for twists with $\|T\| < 1$:

Theorem (Correa da Silva/L 23)

Let T be a crossing-symmetric braided twist compatible with $K \subset H$. Suppose $\|T\| < 1$ and that $\Delta_H^{1/4}|_K$ is not compact. Then

$$\mathcal{L}_T(K)' \cap \mathcal{L}_T(H) = \mathbb{C}1.$$

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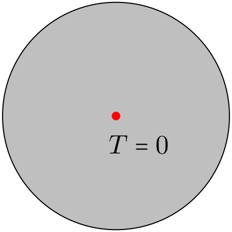

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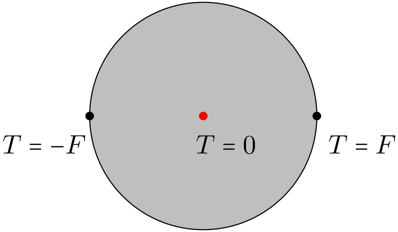

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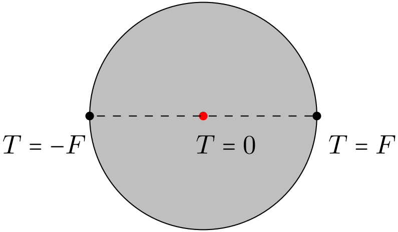

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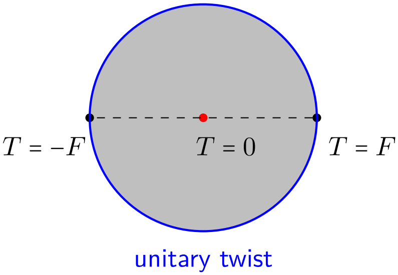

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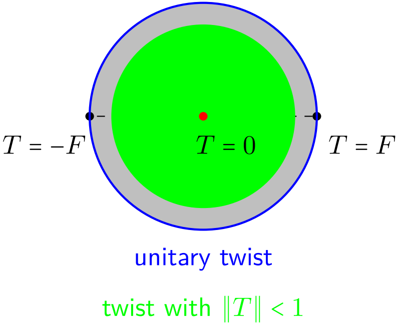

- Local QFT seems to require $\|T\| = 1$ (not fully settled yet).

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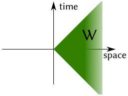
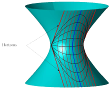
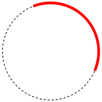
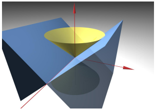
- Approach not restricted to Minkowski space. → Models on deSitter space, the real line, the circle (CFT), higher dimensions ..



Interaction and integrability

Theorem

In any of the situations where modular nuclearity holds (on $\mathbb{R}^{1+1}$), one gets a QFT model satisfying all basic requirements:

- *Locality*
- *Covariance*
- *Positivity of the energy*
- *Reeh-Schlieder property of the vacuum*
- *...*

What is the physical meaning of the twist T?

Theorem

In a model based on wedge-localized H and unitary twist $T = T_S$, one may do (Haag-Ruelle) **scattering theory**.

- *The two-particle S-matrix is elastic and given by $S(\theta_1 - \theta_2)$.*
- *The full S-matrix can be computed and factorizes.*
- *A proof of* **asymptotic completeness** *can be given.*

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- *Locality*
- *Covariance*
- *Positivity of the energy*
- *Reeh-Schlieder property of the vacuum*
- *...*

What is the physical meaning of the twist T?

Theorem

In a model based on wedge-localized H and unitary twist $T = T_S$, one may do (Haag-Ruelle) **scattering theory***.*

- *The two-particle S-matrix is elastic and given by $S(\theta_1 - \theta_2)$.*
- *The full S-matrix can be computed and factorizes.*
- *A proof of* **asymptotic completeness** *can be given.*

- This structure is familiar from integrable models!

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- For various twists, the construction is not yet fully understood:
  - higher dimensions
  - $\|T\| < 1$, we get QFT models based on braid group representations .. interpretation?