Noncommutative Cartan C^*-subalgebras

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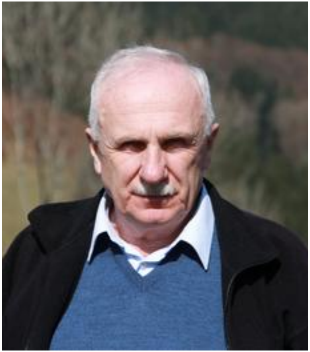
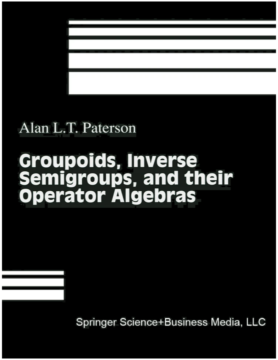


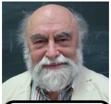
BKK

Ralf Meyer



History

- 1940 Von Neumann $L^\infty(X,\mu) \rtimes G$ are factors for free group actions



- 1971 Vershik 1977 Feldman-Moore

  Cartan W^***-subalgebra** $L^\infty(X,\mu) \subseteq \mathcal{M} \iff$ measure equivalence relation on (X,μ) with a twist
- 1986 Kumjian: C^*-diagonal $C_0(X) \subseteq B \iff$ principle étale locally compact Hausdorff groupoid with a twist
- 2008 Renault: **Cartan** C^***-subalgebra** $C_0(X) \subseteq B \iff$ topologically principle étale Hausdorff groupoids with a twist
- 2011 Exel: **Noncommutative Cartan** C^*-subalgebra $A \subseteq B$ $\implies$ Fell bundle over an inverse semigroup
- 2020 BKK, Meyer: Noncommutative Cartan $A \subseteq B \iff$ closed purely outer Fell bundle over an inverse semigroup

For an inclusion of C^*-algebras $A \subseteq B$ we put

$$N(A) := \{b \in B : bAb^* \subseteq A, \quad b^*Ab \subseteq A\} \qquad \textbf{normalizers}$$

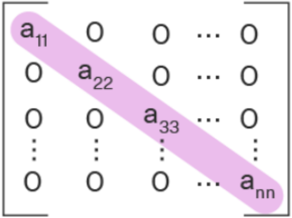
We say that $A \subseteq B$ is **regular** if $B = \overline{span}\, N(A)$ and $AB = B$.

Def. Inclusion of C^*-algebras $A \subseteq B$ is **Cartan** $\Longleftrightarrow$ it is regular, A is **maximal abelian** in B and there is a faithful conditional expectation $E : B \to A$.

Kumjian

Renault

Ex. Diagonal matrices $A \cong \mathbb{C}^n$ are Cartan in $B = M_n(\mathbb{C})$



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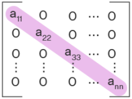
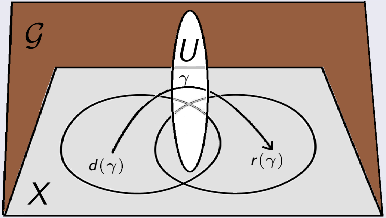


Ex. For a discrete group action G on a loc. compact Hausdorff X

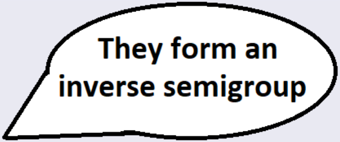
$$C_0(X) \subseteq C_0(X) \rtimes_r G \text{ is Cartan} \iff \begin{array}{c} C_0(X) \text{ is maximal abelian} \\ \text{in } C_0(X) \rtimes_r G \end{array}$$

$\Longleftrightarrow$ the action is topologically free

Let $\mathcal{G}$ be a locally compact Hausdorff groupoid with unit space X, s.t. range, domain $r, d : \mathcal{G} \to X \subseteq \mathcal{G}$ are local homeomorphisms (**étale**).



Bisections $\mathrm{Bis}(\mathcal{G})$ cover the whole $\mathcal{G}$



Then $C_c(\mathcal{G}) = \mathrm{span}\{f \in C_c(U) : U \in \mathrm{Bis}(\mathcal{G})\}$ is a $*$-algebra where

$$(f * g)(\gamma) := \sum_{r(\eta)=r(\gamma)} f(\eta) \cdot g(\eta^{-1} \cdot \gamma), \qquad (f^*)(\gamma) := \overline{f(\gamma^{-1})}.$$

The **reduced C^*-algebra** $C^*_r(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ s.t. $C_c(\mathcal{G}) \ni f \mapsto f|_X \in C_c(X)$ extends to a faithful $E : C^*_r(\mathcal{G}) \to C_0(X)$.

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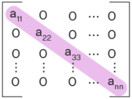
The **reduced C^*-algebra** $C_r^*(\mathcal{G})$ is the completion of $C_c(\mathcal{G})$ s.t. $C_c(\mathcal{G}) \ni f \mapsto f|_X \in C_c(X)$ extends to a faithful $E : C_r^*(\mathcal{G}) \to C_0(X)$.

Ex. If $\mathcal{G} = X \times X$ is the full equivalence relation on $X = \{1, \dots, n\}$,

$$(x, y) \cdot (y, z) = (x, z), \qquad (x, y)^{-1} = (y, x),$$

then $C(X) \cong \mathbb{C}^n$ are diagonal matrices in $C_r^*(\mathcal{G}) \cong M_n(\mathbb{C})$.



Thm. (Renault 2008)

$A \subseteq B$ is a Cartan inclusion $\iff A \cong C_0(X)$ and $B \cong C_r^*(\mathcal{G}, \Sigma)$ for a topologicaly free twisted étale LCH groupoid $(\mathcal{G}, \Sigma)$.

Lem. (A possibly noncommutative)

Any regular C^*-inclusion $A \subseteq B$ has an **inverse semigroup grading**

$$S := \{M \subseteq N(A) : M \text{ is a closed linear space } AM \subseteq M, MA \subseteq M\}$$

with operations inherited from B is an inverse semigroup with unit A:

$$M, N \in S \implies \underbrace{MN}_{\text{closed span}} \in S, \qquad M^* \in S$$

$$(MM^*M = M, \quad M^*MM^* = M^*) \quad \text{and} \quad B = \overline{\sum_{M \in S} M}.$$

J.M.G. Fell

Def. Let S be an **inverse semigroup** with unit 1.

A **Fell bundle** over S is a family $\mathcal{B} = \{B_g\}_{g \in S}$ of Banach spaces equipped with multiplications $B_g \times B_h \to B_{gh}$, involutions $B_g \to B_g^*$ and inclusions $B_g \hookrightarrow B_h$ for $g \leqslant h$, maps satisfying natural axioms.

Then $A := B_1$ is a C^*-algebra, each B_g is a (Hilbert) A-bimodule. They embed into the **reduced C^*-algebra** $C_r^*(\mathcal{B})$ forming S-grading.

Rem. Consider a C^*-inclusion $A \subseteq B$ (A possibly noncommutative)

1. $A' \cap B \subseteq A$ and ~~A is commutative~~ $\Longleftrightarrow$ $A \subseteq B$ is ~~maximal abelian~~
2. every $v \in A' \cap B$ defines an A-bimodule map $A \ni a \mapsto a \cdot v \in B$

Def. A **virtual commutant** is an A-bimodule map $I \to B$ defined on $I \lhd A$. Regular C^*-inclusion $A \subseteq B$ is **noncommutative Cartan** $\Longleftrightarrow$ any virtual commutant has range in A and there is a faithful conditional expectation $E \colon B \to A$.


Ruy Exel

Thm. (Exel 2011)

$A \subseteq B$ is noncommutative Cartan $\Longrightarrow$ $B \cong C^*_r(\mathcal{B})$ for a Fell bundle

$$\mathcal{B} = \{B_g\}_{g \in S} \text{ with } B_1 = A.$$

Question:

Which Fell bundles $\mathcal{B} = \{B_g\}_{g \in S}$ give Cartan inclusions?

Def. (BKK, Meyer) Let $\mathcal{B} = \{B_g\}_{g \in S}$ be Fell bundle over S, $A := B_1$

$\mathcal{B}$ is **closed** if $B_g = B_{g,1} \oplus B_{g,1}^{\perp}$ as A-bimodules for $B_{g,1} := B_g \cap A$, $g \in S$

$\mathcal{B}$ is **purely outer** if $B_{g,1}^{\perp}$ is purely outer as an A-bimodule, $g \in S$.

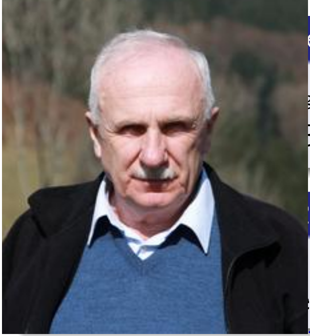
Thm. (BKK, Meyer 2020)

$A \subseteq B$ is noncommutative Cartan $\iff$ $B \cong C_r^*(\mathcal{B})$ and $A = B_1$ for a closed, purely outer Fell bundle $\mathcal{B} = \{B_g\}_{g \in S}$ (unique after refinement)

Cor. Let $A \subseteq B$ be a regular C^*-inclusion with A **simple**. TFAE:

1. $A \subseteq B$ is a **noncommutative Cartan subalgebra**
2. $B \cong \mathrm{C}_r^*(\mathcal{B})$ for an **outer Fell bundle** $\mathcal{B} = (B_g)_{g \in G}$ over a **group** G with the unit fiber $B_1 = A$ ($\mathcal{B}$ and G are uniquely determined)
3. $A \subseteq B$ is a **C*****-irreducible**, i.e. all intermediate $A \subseteq C \subseteq B$ are simple

Ex. The **CAR algebra** M_{2^∞} is a noncommutative Cartan subalgebra of the **Cuntz algebra** $\mathcal{O}_2 = C^*(S_1, S_2 \text{ isometries} : S_1 S_1^* + S_2 S_2^* = 1)$



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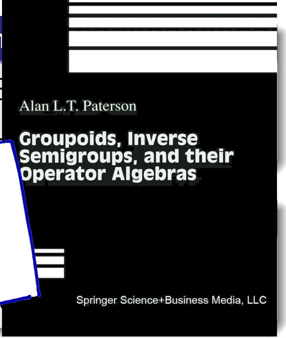

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