Quasiderivations and Quantum Mishchenko-Fomenko Construction

Yasushi Ikeda

Sapporo, Japan

July 6, 2023

WGMP

Outline

Poisson Bracket on the Dual Space of a Lie Algebra

- We are going to investigate a quantum analogue of the theorem of A. Mishchenko and A. Fomenko.
- The dual space g^* of a finite dimensional real Lie algebra g is a Poisson manifold and the following diagram commutes. The Poisson bracket is called the Kirillov-Kostant bracket.

$$
\begin{array}{ccc}
C^\infty(g^*) \otimes C^\infty(g^*) & \xrightarrow{\text{Poisson bracket}} & C^\infty(g^*) \\
\uparrow & & \uparrow \\
g \otimes g & \xrightarrow[\text{Lie bracket}]{} & g
\end{array}
$$

Classical Theorem of A. Mishchenko and A. Fomenko

The classical theorem of A. Mishchenko and A. Fomenko is the following.[1]

Theorem (A. Mishchenko and A. Fomenko, 1978)

Suppose that ∂_ξ is a constant vector field on the dual space g^*. We have

$$\{\partial_\xi^m(x), \partial_\xi^n(y)\} = 0$$

for any m and n and for any Poisson central elements x and y of the symmetric algebra $S(g)$.

[1] Mishchenko and Fomenko, "Euler equations on finite-dimensional Lie groups".

Classical Theorem of A. Mishchenko and A. Fomenko

We are going to investigate a quantum analogue of this theorem.

1. The symmetric algebra $S(g)$ should be replaced by the universal enveloping algebra $U(g)$.
2. The Poisson bracket should be replaced by the commutator on the universal enveloping algebra $U(g)$.
3. We need to find a “derivation” of the universal enveloping algebra $U(g)$.

Quantum Analogue of A. Mishchenko and A. Fomenko

We consider $g = gl(d, \mathbb{C})$.

- Let
$$e = \begin{pmatrix} e_1^1 & \cdots & e_d^1 \\ \vdots & \ddots & \vdots \\ e_1^d & \cdots & e_d^d \end{pmatrix} \in M\big(d, gl(d, \mathbb{C})\big),$$
where e_j^i form a linear basis of $gl(d, \mathbb{C})$ and satisfy the commutation relations $\left[e_j^i, e_l^k\right] = e_j^k \delta_l^i - \delta_j^k e_l^i$.
- A constant vector field on the dual space is given by
$$\partial_\xi = \operatorname{tr}(\xi \partial), \qquad \partial_j^i = \frac{\partial}{\partial e_i^j}$$
where ξ is a numerical matrix.

Quantum Analogue of A. Mishchenko and A. Fomenko

Remark

The derivation

$$Sgl(d,\mathbb{C}) \to M(d, Sgl(d,\mathbb{C})), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. We have $\partial \nu = 0$ for any scalar ν.
2. We have $\partial\, \mathrm{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. We have the Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y)$$

for any elements x and y of the symmetric algebra.

Quantum Analogue of A. Mishchenko and A. Fomenko

- There is no such a derivation on the universal enveloping algebra $Ugl(d, \mathbb{C})$ since we obtain a contradiction

$$
\begin{aligned}
0 &= \partial\left(e^i_j e^k_l - e^k_l e^i_j\right) && \text{(Leibniz rule)} \\
&= \partial\left(e^k_j \delta^i_l - \delta^k_j e^i_l\right) && \text{(commutation relation)} \\
&= \delta_j \delta^k \delta^i_l - \delta^k_j \delta_l \delta^i && \text{(second conditon)} \\
&\neq 0
\end{aligned}
$$

  if such a derivation ∂ exists.

- Gurevich, Pyatov, and Saponov defined the quasiderivation of the universal enveloping algebra.

Quasiderivation of $Ugl(d, \mathbb{C})$

Definition (Gurevich, Pyatov, and Saponov, 2012)

The quasiderivation

$$Ugl(d, \mathbb{C}) \to M(d, Ugl(d, \mathbb{C})), \qquad x \mapsto \partial x$$

is a unique linear mapping satisfying the following.

1. We have $\partial \nu = 0$ for any scalar ν.
2. We have $\partial \operatorname{tr}(\xi e) = \xi$ for any numerical matrix ξ.
3. We have the twisted Leibniz rule

$$\partial(xy) = (\partial x)y + x(\partial y) + (\partial x)(\partial y)$$

for any elements x and y of the universal enveloping algebra.

Conjecture (Quantum Analogue)

Conjecture

Suppose that ξ is a numerical matrix and let $\partial_\xi = \mathrm{tr}(\xi\partial)$. We have

$$\left[\partial_\xi^m(x), \partial_\xi^n(y)\right] = 0$$

for any m and n and for any central elements x and y of the universal enveloping algebra $Ugl(d, \mathbb{C})$.

Recently I and Georgiy Sharygin believe that we successfully proved this conjecture and we are preparing.

Fundamental Formula and Corollary

We assume the following form

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(g_m^{(n-1)}(e)_j (e^m)^i + h_m^{(n-1)}(e)(e^m)^i_j \right),$$

where $g_m^{(n-1)}$ and $h_m^{(n-1)}$ are polynomials.

Fundamental Formula and Corollary

We have

$$\begin{aligned}
\partial(e^{n+1})^i_j &= \sum_{k=1}^{d} \partial\Big((e^n)^i_k e^k_j\Big) \\
&= \sum_{k=1}^{d} \Big(\partial(e^n)^i_k e^k_j + (e^n)^i_k \partial e^k_j + \partial(e^n)^i_k \partial e^k_j\Big) \\
&= \sum_{k=1}^{d} \partial(e^n)^i_k e^k_j + \boxed{\delta_j(e^n)^i} + \sum_{k=1}^{d} \partial(e^n)^i_k \delta_j \delta^k
\end{aligned}$$

by the twisted Leibniz rule.

Fundamental Formula and Corollary

We compute the first term. We have

$$\sum_{k=1}^{d}\sum_{m=0}^{n-1} g_m^{(n-1)}(e)_k (e^m)^i e_j^k - \boxed{\sum_{m=0}^{n-1}\left(g_m^{(n-1)}(e)e\right)_j (e^m)^i}$$

$$= \sum_{k=1}^{d}\sum_{m=0}^{n-1} g_m^{(n-1)}(e)_k \left[(e^m)^i, e_j^k\right]$$

$$= \sum_{k=1}^{d}\sum_{m=0}^{n-1} g_m^{(n-1)}(e)_k \left((e^m)^k \delta_j^i - \delta^k (e^m)_j^i\right)$$

$$= \sum_{m=0}^{n-1}\left(\boxed{g_m^{(n-1)}(e)e^m\delta_j^i} - g_m^{(n-1)}(e)(e^m)_j^i\right) \tag{1}$$

by the commutation relation.

Fundamental Formula and Corollary

We have

$$\sum_{k=1}^{d}\sum_{m=0}^{n-1} h_m^{(n-1)}(e)(e^m)^i_k e^k_j = \boxed{\sum_{m=0}^{n-1} h_m^{(n-1)}(e)(e^{m+1})^i_j.}$$

We compute the third term. We have

$$\sum_{k=1}^{d}\sum_{m=0}^{n-1}\left(g_m^{(n-1)}(e)_k(e^m)^i + h_m^{(n-1)}(e)(e^m)^i_k\right)\delta_j\delta^k$$

$$= \sum_{m=0}^{n-1}\left(g_m^{(n-1)}(e)(e^m)^i_j + \boxed{h_m^{(n-1)}(e)_j(e^m)^i}\right). \quad (2)$$

Fundamental Formula and Corollary

The second term of the equation (1) and the first term of the equation (2) are cancelled out and we have

$$\begin{aligned}\partial(e^{n+1})^i_j &= \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)e + h_m^{(n-1)}(e)\Big)_j (e^m)^i + \delta_j (e^n)^i \\ &\quad + \sum_{m=0}^{n-1}\Big(g_m^{(n-1)}(e)e^m\delta^i_j + h_m^{(n-1)}(e)(e^{m+1})^i_j\Big) \\ &= \sum_{m=0}^{n}\Big(g_m^{(n)}(e)_j (e^m)^i + h_m^{(n)}(e)(e^m)^i_j\Big).\end{aligned}$$

Fundamental Formula and Corollary

We obtained the recursion formulae

1. $g_m^{(n)}(x) = g_m^{(n-1)}(x)x + h_m^{(n-1)}(x)$ for $0 \leq m < n$
2. $g_n^{(n)}(x) = 1$ for $0 \leq n$
3. $h_0^{(n)}(x) = \sum_{m=0}^{n-1} g_m^{(n-1)}(x)x^m$ for $0 \leq n$
4. $h_m^{(n)}(x) = h_{m-1}^{(n-1)}(x)$ for $0 < m \leq n$

and the solutions to them are

$$g_m^{(n)}(x) = f_+^{(n-m)}(x), \qquad h_m^{(n)}(x) = f_-^{(n-m)}(x),$$

where we define the polynomials

$$f_\pm^{(n)}(x) = \frac{(x+1)^n \pm (x-1)^n}{2} = \sum_{m=0}^{n} \frac{1 \pm (-1)^{n-m}}{2} \binom{n}{m} x^m.$$

Fundamental Formula and Corollary

We obtained a fundamental theorem for quasiderivations of central elements.

Theorem (I, 2022)

We have the formula

$$\partial(e^n)^i_j = \sum_{m=0}^{n-1} \left(f_+^{(n-m-1)}(e)_j (e^m)^i + f_-^{(n-m-1)}(e)(e^m)^i_j \right)$$

for any nonnegative integer n.

Fundamental Formula and Corollary

The center of the universal enveloping algebra

$$\mathcal{Z}\left(U\mathfrak{gl}(d,\mathbb{C})\right) \simeq \mathbb{C}\left[(\operatorname{tr} e^n)_{n=1}^d\right]$$

is a free commutative algebra on the set $\{\operatorname{tr} e^n\}_{n=1}^d$.

Corollary

The conjecture holds for $m = n = 1$. We have

$$\left[\partial_\xi(x), \partial_\xi(y)\right] = 0$$

for any central elements x and y.

Generators of Second-Order Quasiderivations

According to the theorem the second quasiderivations

$$\partial_\xi^2(\operatorname{tr} e^n), \qquad n = 0, 1, \dots$$

are spanned over the center by the elements

$$\operatorname{tr}\big(\xi\partial \operatorname{tr}(\xi e^{n+m})e^n\big) + \operatorname{tr}\big(\xi\partial \operatorname{tr}(\xi e^n)e^{n+m}\big), \quad m, n = 0, 1, \dots$$

We have the following theorem.

Generators of Second-Order Quasiderivations

Theorem (I, 2023)

We have

$$\operatorname{span}\Big\{\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^{n+2m})e^n\big)+\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^n)e^{n+2m}\big)\Big\}_{m,n=0}^{\infty}$$
$$=\operatorname{span}\Big\{\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^n)e^n\big)\Big\}_{n=0}^{\infty},$$
$$\operatorname{span}\Big\{\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^{n+2m+1})e^n\big)+\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^n)e^{n+2m+1}\big)\Big\}_{m,n=0}^{\infty}$$
$$=\operatorname{span}\Big\{\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^{n+1})e^n\big)+\operatorname{tr}\big(\xi\partial\operatorname{tr}(\xi e^n)e^{n+1}\big)\Big\}_{n=0}^{\infty}$$

up to the subspace generated by the set $\{\operatorname{tr}(\xi e^i)\operatorname{tr}(\xi e^j)\}_{i,j=0}^{\infty}$.

Generators of Second-Order Quasiderivations

Theorem (I, 2023)

We have

$$\operatorname{span}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n+\cancel{2m}})e^{n}\right)+\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+\cancel{2m}}\right)\right\}_{\cancel{m},n=0}^{\infty}$$

$$=\operatorname{span}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n})e^{n}\right)\right\}_{n=0}^{\infty},$$

$$\operatorname{span}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n+\cancel{2m}+1})e^{n}\right)+\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+\cancel{2m}+1}\right)\right\}_{\cancel{m},n=0}^{\infty}$$

$$=\operatorname{span}\left\{\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n+1})e^{n}\right)+\operatorname{tr}\left(\xi\partial\operatorname{tr}(\xi e^{n})e^{n+1}\right)\right\}_{n=0}^{\infty}$$

up to the subspace generated by the set $\left\{\operatorname{tr}(\xi e^{i})\operatorname{tr}(\xi e^{j})\right\}_{i,j=0}^{\infty}$.

Key Matrix and Symmetry

Definition

1. We define P_n as the n by n submatrix of the following matrix.

$$\begin{pmatrix} \vdots \\ f_+^{(4)}(x) \\ f_+^{(3)}(x) \\ f_+^{(2)}(x) \\ f_+^{(1)}(x) \\ f_+^{(0)}(x) \end{pmatrix} = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 6 & 0 & 1 & \cdots \\ 0 & 3 & 0 & 1 & 0 & \cdots \\ 1 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \\ x^4 \\ \vdots \end{pmatrix}$$

P_5

2. We define $P_n^{(m)}$ as the matrix P_n shifted to the right by m positions.

Key Matrix and Symmetry

We have

$$\operatorname{tr}\bigl(\xi\partial\operatorname{tr}(\xi e^m)e^n\bigr) = \operatorname{tr}\left(\begin{pmatrix}\xi & \xi e & \cdots & \xi e^{m+n-1}\end{pmatrix} P_m^{(n)} \begin{pmatrix}\xi \\ \xi e \\ \vdots \\ \xi e^{m+n-1}\end{pmatrix}\right).$$

Key Matrix and Symmetry

Suppose that A is a numerical square matrix.

Definition

We define

$$\tau(A) = \begin{pmatrix} A_1^1 & A_2^1 + A_1^2 & \cdots & A_n^1 + A_1^n \\ 0 & A_2^2 & \cdots & A_n^2 + A_2^n \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n^n \end{pmatrix}.$$

Key Matrix and Symmetry

We have

$$\operatorname{tr}\left(\begin{pmatrix}\xi & \xi e & \cdots & \xi e^{n-1}\end{pmatrix} A \begin{pmatrix}\xi \\ \xi e \\ \vdots \\ \xi e^{n-1}\end{pmatrix}\right) = \operatorname{tr}\left(\begin{pmatrix}\xi & \xi e & \cdots & \xi e^{n-1}\end{pmatrix} \tau(A) \begin{pmatrix}\xi \\ \xi e \\ \vdots \\ \xi e^{n-1}\end{pmatrix}\right)$$

since we have $\operatorname{tr}(\xi e^i \xi e^j) = \operatorname{tr}(\xi e^j \xi e^i)$.

Main Theorem (Matrix Form)

Theorem (I, 2023)

We have

$$\tau \begin{pmatrix} 0 & P_{n+2m} \\ P_n^T & 0 \end{pmatrix} = \sum_{\ell=0}^{m} \left(\binom{2m-\ell}{\ell} + \binom{2m-\ell-1}{\ell-1} \right) P_{n+\ell}^{(n+\ell)}$$

and

$$\tau \begin{pmatrix} 0 & P_{n+2m+1} \\ P_n^T & 0 \end{pmatrix} = \sum_{\ell=0}^{m} \binom{2m-\ell}{\ell} \left(P_{n+\ell+1}^{(n+\ell)} + P_{n+\ell}^{(n+\ell+1)} \right)$$

for any nonnegative integers m and n.

Main Theorem (Matrix Form)

We would like to expand $\tau(P_{2m})$ along $(\ell, \ell+1)$ elements.

$$\tau(P_{2m}) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 4 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & 0 & 11 & 0 & 2 \\ 0 & 0 & 9 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \dots$$

$$= 2P_1^{(1)}, 4P_1^{(1)} + 2P_2^{(2)}, 6P_1^{(1)} + 9P_2^{(2)} + 2P_3^{(3)}, \dots$$

$$= \sum_{\ell=1}^{m} \left(\binom{2m-\ell}{\ell} + \binom{2m-\ell-1}{\ell-1} \right) P_\ell^{(\ell)}.$$

Main Theorem (Matrix Form)

Let $(m, n) = (2, 1)$. We have

$$\tau \begin{pmatrix} 0 & P_{n+2m} \\ P_n^T & 0 \end{pmatrix} = \tau \begin{pmatrix} 0 & P_5 \\ P_1^T & 0 \end{pmatrix} = \tau \begin{pmatrix} 0 & 1 & 0 & 6 & 0 & 1 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 6 & 0 & 2 \\ 0 & 0 & 4 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 1P_1^{(1)} + 4P_2^{(2)} + 2P_3^{(3)}.$$

$\tau(P_{2m})$

Main Theorem (Matrix Form)

We have

$$\tau \begin{pmatrix} 0 & P_{n+2m} \\ P_n^T & 0 \end{pmatrix} = \sum_{\ell=0}^{m} \left(\binom{2m-\ell}{\ell} + \binom{2m-\ell-1}{\ell-1} \right) P_{n+\ell}^{(n+\ell)}.$$

Equivalent Condition (Even Case)

The first part of the theorem is equivalent to the following.

1. We have

$$\binom{2n_1+n_2+2n_3+1}{2n_3}+\binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3}\left(\binom{n_1+n_2+n_3+n_4+1}{2n_4}+\binom{n_1+n_2+n_3+n_4}{2n_4}\right) \times \binom{n_1+n_3-n_4}{2(n_3-n_4)}$$

for any nonnegative integers $(n_k)_{k=1}^{3}$.

Equivalent Condition (Even Case)

2 We have

$$f_+^{(n+2m)}(x) + x^{2m} f_+^{(n)}(x) = \sum_{\ell=0}^{m} \left(\binom{2m-\ell}{\ell} + \binom{2m-\ell-1}{\ell-1} \right) x^\ell f_+^{(n+\ell)}(x)$$

for any nonnegative integer n.

$$f_+^{(n)}(x) = \frac{(x+1)^n + (x-1)^n}{2}.$$

These conditions have been verified using Mathematica.

Equivalent Condition (Odd Case)

The second part of the theorem is equivalent to the following.

1. We have

$$\binom{2n_1+n_2+2n_3+2}{2n_3}+\binom{n_2+2n_3}{2n_3} = \sum_{n_4=0}^{n_3}\binom{n_1+n_2+n_3+n_4+1}{2n_4} \times\left(\binom{n_1+n_3-n_4+1}{2(n_3-n_4)}+\binom{n_1+n_3-n_4}{2(n_3-n_4)}\right)$$

for any nonnegative integers $(n_k)_{k=1}^{3}$.

Equivalent Condition (Odd Case)

2. We have

$$f_+^{(n+2m+1)}(x) + x^{2m+1} f_+^{(n)}(x) = \sum_{\ell=0}^{m} \binom{2m-\ell}{\ell} \left(x^{\ell} f_+^{(n+\ell+1)}(x) + x^{\ell+1} f_+^{(n+\ell)}(x) \right)$$

for any nonnegative integer n.

$$f_+^{(n)}(x) = \frac{(x+1)^n + (x-1)^n}{2}.$$

These conditions have been verified using Mathematica.

Conclusion

- In a quantum analogue of the theorem of A. Mishchenko and A. Fomenko, the derivation of the symmetric algebra $Sgl(d, \mathbb{C})$ is replaced by the quasiderivation of the universal enveloping algebra $Ugl(d, \mathbb{C})$.
- We derived a concrete formula and proved the quantum analogue for order 1. Higher quasiderivations can be computed using this formula as well.
- I and Georgiy Sharygin believe that we successfully proved the quantum analogue. We are preparing a paper.
- We succeeded to reduce the number of the generators of the second quasiderivations. We expect that each higher quasiderivations will be generated by reduced number of generators as well.

References

- Gurevich, Dimitri, Pavel Pyatov, and Pavel Saponov. "Braided Weyl algebras and differential calculus on $U(u(2))$". In: *Journal of Geometry and Physics* 62.5 (2012), pp. 1175–1188.
- Ikeda, Yasushi. "Quasidifferential operator and quantum argument shift method". In: *Theoretical and Mathematical Physics* 212.1 (2022), pp. 918–924.
- Ikeda, Yasushi and Georgiy Sharygin. "The argument shift method in universal enveloping algebra $U\mathfrak{gl}_d$ (in preparation)". In: ().
- Mishchenko, AS and AT Fomenko. "Euler equations on finite-dimensional Lie groups". In: *Mathematics of the USSR-Izvestiya* 12.2 (1978), pp. 371–389.
- Sharygin, Georgiy. "Quasi-derivations on $U\mathfrak{gl}_n$ and the argument shift method". In: *Contemporary Mathematics* 789 (2023).