Complete Classification of Local Conservation Laws for Generalized Cahn–Hilliard–Kuramoto–Sivashinsky Equation

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The Kuramoto–Sivashinsky equation is an evolutionary PDE in one dependent variable u and $n+1$ independent variables $t, x_1 \ldots, x_n$ that reads

$$u_t + \Delta^2 u + \Delta u + |\nabla u|^2/2 = 0, \qquad (1)$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator, $|\nabla u|^2 = \sum_{i=1}^{n} (\partial u/\partial x_i)^2$, and n is an arbitrary natural number.

This equation originally came up in physical and chemical contexts and describes inter alia flame propagation, reaction-diffusion processes and unstable drift waves in plasmas.

The Cahn–Hilliard equation is an evolutionary PDE in one dependent variable u and $n+1$ independent variables $t, x_1 \ldots, x_n$ that reads

$$u_t = c_1 \Delta(u^3 - u + c_2 \Delta u), \qquad (2)$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$ is the Laplace operator and c_1, c_2 are constants, and n is an arbitrary natural number. This equation describes e.g. the process of phase separation in a binary alloy, with applications in many areas such as complex fluids, interfacial fluid flow, polymer science, spinodal decomposition and tumor growth simulation.

The generalized CHKS equation

In what follows we shall consider the following *generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation*, a natural generalization of the both original C-H and K-S equations that reads

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u), \tag{3}$$

where a is a nonzero constant, b, f, g are arbitrary smooth real-valued functions of the dependent variable u.

While many authors, see e.g. [1,2,3,4], have considered various generalizations of the Kuramoto-Sivashinsky equation or Cahn-Hilliard equation (1), there is no broadly accepted definition of the term 'generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation'.

[1] Bozhkov Y., Dimas S. Group classification and conservation laws for a two-dimensional generalized Kuramoto-Sivashinsky equation. Nonlinear Anal. 84 (2013), 117–135.

[2] Kudryashov N.A. Exact solutions of the generalized Kuramoto-Sivashinsky equation, Phys. Lett. A 147 (1990), 287–291.

[3] Rosa M., Camacho J.C., Bruzón M., Gandarias M. L. Lie symmetries and conservation laws for a generalized Kuramoto–Sivashinsky equation. Math. Meth. Appl. Sci. 41 (2018), no. 17, 7295–7303.

[4] Cherfils L., Miranvill, A., Zelik S. On a generalized Cahn-Hilliard equation with biological applications. Dis. & Cont. Dyn. Sys. - B, 19(7), 2013–2026. (2014)

The generalized CHKS equation II

Our goal is to describe all local conservation laws for (3), i.e.,

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u),$$

To the best of our knowledge, this was not yet done, especially for the arbitrary number n of the space variables, although some partial results[5,6,7] on conservation laws of several different generalizations of the Kuramoto–Sivashinsky equation for $n = 1, 2$ are known.

[5] Bozhkov Y., Dimas S. Group classification and conservation laws for a two-dimensional generalized Kuramoto-Sivashinsky equation. Nonlinear Anal. 84 (2013), 117–135.

[6] Govinder K.S., Narain R., Okeke J.E. New exact solutions and conservation laws of a class of Kuramoto Sivashinsky (KS) equations. Quaest. Math. 42 (2019), no. 1, 93–112

[7] Rosa M., Camacho J.C., Bruzón M., Gandarias M. L. Lie symmetries and conservation laws for a generalized Kuramoto–Sivashinsky equation. Math. Meth. Appl. Sci. 41 (2018), no. 17, 7295–7303.

Why should one care about conservation laws

While the conservation laws originated from physical principles such as conservation of energy, mass and momentum, there are numerous applications of conservation laws beyond that.

For instance, for a given ODE finding conservation laws are equivalent to searching for a constants of a motion, which can then be used for the reduction of the order.

While an infinite number of conservation laws often indicates integrability, even the presence of finitelly many conservation laws can be quite helpful in analyzing behaviour of a solution in terms of existence, uniqueness and stability. One can also improve numerical solving of PDEs using discretizations respecting known conservation laws. Just as importantly, there is a connection between conservation laws and symmetries of a given system through Noether's theorem or its Hamiltonian version in the evolutionary case.

Preliminaries

Mostly following the book P.J. Olver, Applications of Lie groups to differential equations, Springer, 1993, consider a totally nondegenerate system $\mathscr{E}$ of m partial differential equations in $n+1$ independent variables $x_0, x_1 \ldots, x_n$ and m unknown functions $u^1, \ldots, u^m$ of the form

$$F^i(x_j, u^a, \{u_I^b\}) = 0, \; i = 1, \ldots, m$$

where F^i are smooth functions of the variables x_j, u^a, u_I^b, $I = (i_0, i_1, \ldots, i_n) \in \mathbb{Z}_+^{n+1}$ is a multi-index and

$$u_I^b = \frac{\partial^{|I|} u^b}{\partial x_0^{i_0} \partial x_1^{i_1} \ldots \partial x_n^{i_n}}.$$

The **total derivatives** are differential operators

$$D_{x_j} = \frac{\partial}{\partial x_j} + \sum_{b,I} u_{Ij}^b \frac{\partial}{\partial u_I^b},$$

where $Ij = (i_0, \ldots, i_j + 1, \ldots, i_n)$

A **local conservation law** for $\mathscr{E}$ is a differential expression

$$D_{x_0}P_0 + D_{x_1}P_1 + \ldots + D_{x_n}P_n$$

that vanishes modulo $\mathscr{E}$ and its differential consequences. Here P_i depend on x_i, u^j and finitely many derivatives of u^k.

A local conservation law is said to be **trivial** if P itself vanishes modulo $\mathscr{E}$ and its differential consequences or if the expression $D_{x_0}P_0 + D_{x_1}P_1 + \ldots + D_{x_n}P_n$ is identically zero no matter whether $\mathcal{E}$ holds or not.

It is readily seen that a linear combination of two conservation laws for $\mathcal{E}$ is again a conservation law for $\mathcal{E}$, so the conservation laws for a given system $\mathcal{E}$ form a vector space.

Two conservation laws are said to be **equivalent** if they differ by a trivial conservation law.

A conservation law in characteristic form for $\mathcal{E}$ is a conservation law which can be written as

$$D_{x_0}P_0 + D_{x_1}P_1 + \ldots + D_{x_n}P_n = Q \cdot F;$$

then we refer to Q as to the **characteristic** of this conservation law.

The vector function Q is a characteristic of a conservation law if and only if

$$\mathrm{D}_F^*(Q) + \mathrm{D}_Q^*(F) = 0,$$

where

$$(\mathrm{D}_P^*)_{ab} = \sum_I (-D)_I \circ \frac{\partial P_b}{\partial u_I^a}$$

is a differential operator in total derivatives.

Note that a local conservation law is trivial if and only if its characteristic is identically zero.

Main result

Describing all nontrivial local conservation laws of all orders for a given PDE is a difficult problem handled mostly on a case-by-case basis, and we succeeded in addressing this problem for (3).

Theorem 1

Let equation (3), *that is,*

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u),$$

where $\Delta = \sum_{i=1}^{n} \partial^2/\partial x_i^2$, $|\nabla u|^2 = \sum_{i=1}^{n}(\partial u/\partial x_i)^2$ *and* n *is a natural number, satisfy one of the following conditions:*

1. $f \neq \dfrac{\partial b}{\partial u}$,
2. $f = 0$, $\dfrac{\partial^2 g}{\partial u^2} \neq 0$ *and* b *is constant,*
3. $f = \dfrac{\partial b}{\partial u} \neq 0$, $\dfrac{\partial^2 g}{\partial u^2} \neq 0$ *and* $\dfrac{\partial^3 g}{\partial u^3}\dfrac{\partial b}{\partial u} - \dfrac{\partial^2 g}{\partial u^2}\dfrac{\partial^2 b}{\partial u^2} \neq 0.$

Then equation (3) *admits no nontrivial local conservation laws.*

Main result II

In particular, the above theorem implies that the original Kuramoto–Sivashinsky equation (1), i.e.,

$$u_t + \Delta^2 u + \Delta u + |\nabla u|^2/2 = 0,$$

has no nontrivial local conservation laws of any order for any $n = 1, 2, 3, \ldots$

We now turn to the cases when (3) *does* admit nontrivial conservation laws and list below just the densities of conservation laws of the equation in question up to an obvious equivalence. The associated fluxes are too cumbersome to list here and can be found in the paper P. Holba, *Stud Appl Math.*, to appear, https://doi.org/10.1111/sapm.12576.

It is tacitly assumed here and below that the densities of all nontrivial local conservation laws are given modulo the addition of densities of trivial local conservation laws.

Main result III

Theorem 2a

Let $f = 0$, $b = c_1$ *and* $g = c_2 u + c_3$,*where* c_1, c_2, c_3 *are constants. Then the densities of all nontrivial local conservation laws for equation* (3), *which in this case is just a linear inhomogeneous PDE of the form*

$$u_t = a\Delta^2 u + c_1 \Delta u + c_2 u + c_3,$$

read

$$T^{\mathrm{I}} = uQ^{\mathrm{I}} \quad (4)$$

where $Q^{\mathrm{I}} = Q^{\mathrm{I}}(t, x_1, \ldots, x_n)$ *is any (smooth) solution of the linear PDE with constant coefficients*

$$\frac{\partial Q^{\mathrm{I}}}{\partial t} + a\Delta^2(Q^{\mathrm{I}}) + c_2 Q^{\mathrm{I}} + c_1 \Delta(Q^{\mathrm{I}}) = 0. \quad (5)$$

Main result IV

Theorem 2b

Let $f = \partial b/\partial u \neq 0$ and $g = c_2 u + c_3$, where c_2 and c_3 are constants. Then the densities of all nontrivial local conservation laws for (3), which now can be written as

$$u_t = a\Delta^2 u + \Delta(\tilde{b}(u)) + c_2 u + c_3,$$

where $\tilde{b}$ is such that $\partial\tilde{b}/\partial u = b$, have the form

$$T^{\mathrm{II}} = \tilde{Q}^{\mathrm{II}} e^{-c_2 t} u \qquad (6)$$

and $\tilde{Q}^{\mathrm{II}} = \tilde{Q}^{\mathrm{II}}(x_1, \dots, x_n)$ is any (smooth) solution of the linear Laplace equation $\Delta\tilde{Q}^{\mathrm{II}} = 0$.

Note that for $c_2 = c_3 = 0$ and $\tilde{b} = k(u^3 - u)$, where k is an arbitrary nonzero constant, this gives a complete description of local conservation laws for the original Cahn–Hilliard equation.

Main result V

Theorem 2c

Let $f = \partial b/\partial u \neq 0$, $\partial^2 g/\partial u^2 \neq 0$ *and* $\dfrac{\partial^3 g}{\partial u^3}\dfrac{\partial b}{\partial u} = \dfrac{\partial^2 g}{\partial u^2}\dfrac{\partial^2 b}{\partial u^2}$.

Then there exist such constants $c_4 \neq 0$ *and* c_5 *that* $b = (1/c_4)\partial g/\partial u + c_5$, *and the densities of all nontrivial local conservation laws for* (3), *which now can be written as*

$$u_t = a\Delta^2 u + \Delta(\tilde{b}(u)) + g(u),$$

where $\tilde{b} = (1/c_4)g + c_5 u$, *have the form*

$$T^{\mathrm{III}} = e^{(c_4 c_5 - a c_4^2)t} u \tilde{Q}^{\mathrm{III}} \quad (7)$$

where $\tilde{Q}^{\mathrm{III}} = \tilde{Q}^{\mathrm{III}}(x_1, \ldots, x_n)$ *is any (smooth) solution of a linear PDE with constant coefficients*

$$\Delta\tilde{Q}^{\mathrm{III}} + c_4\tilde{Q}^{\mathrm{III}} = 0. \quad (8)$$

Outline of proof

To prove the above results, we first recall that (3) has the form

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u),$$

and a **local conservation law** for (3) is a differential expression

$$D_t(T) + D_{x_1}(P_1) + \ldots + D_{x_n}(P_n)$$

that vanishes modulo (3) and its differential consequences. Here T and P_i depend on t, x_i, u and finitely many derivatives of u.

Since (3) is an evolution equation we can without loss of generality assume that T and P_i depend only on t,x_i, u and finitely many x-derivatives of u but do not depend on t-derivatives of u and derivatives of u involving both t and x_i, cf. e.g. P.J. Olver, Applications of Lie groups to differential equations, Springer, 1993.

Outline of proof II

The necessary condition for a differential function Q to be a characteristics of conservation law for equation (3) reads

$$D_t(Q) + a\sum_{i,j}^{n} D_{x_i x_i x_j x_j}(Q) + \left(2\frac{\partial b}{\partial u} - 2f\right)\sum_{i=1}^{n}\left[u_{x_i x_i}Q + u_{x_i}D_{x_i}(Q)\right] +$$
$$+\left(\frac{\partial^2 b}{\partial u^2} - \frac{\partial f}{\partial u}\right)\sum_{i=1}^{n} u_{x_i}^2 Q + \frac{\partial g}{\partial u}Q + b\sum_{i=1}^{n} D_{x_i x_i}(Q) = 0.$$

Equation (3) satisfies the conditions of Theorem 6 from the paper S. A. Igonin, Conservation laws for multidimensional systems and related linear algebra problems, J. Phys. A: Math. Gen. 35 (2002), no. 49, 10607–10617, and hence a characteristic Q of a local conservation law for (3) can depend at most on $t, x_1, \ldots, x_n$.

Therefore without loss of generality we can assume that the density T of the associated conservation law depends at most on $t, x_1, \ldots, x_n$ and u, and we have $\partial T/\partial u = Q$.

Outline of proof III

With this in mind it is readily verified that Q is a characteristic of conservation law for (3) if and only if it satisfies the following equation:

$$\frac{\partial Q}{\partial t} + a\sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + b\sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} + \\
+ \left(2\frac{\partial b}{\partial u} - 2f\right) \sum_{i=1}^{n} \left(u_{x_i x_i} Q + u_{x_i} \frac{\partial Q}{\partial x_i}\right) + \\
+ \left(\frac{\partial^2 b}{\partial u^2} - \frac{\partial f}{\partial u}\right) \sum_{i=1}^{n} u_{x_i}^2 Q + \frac{\partial g}{\partial u} Q = 0. \tag{9}$$

Outline of proof IV

As Q is independent of u and its derivatives, applying $\partial/\partial u_{x_i x_i}$ to (9) for any i we get

$$\left(2\frac{\partial b}{\partial u} - 2f\right) Q = 0, \tag{10}$$

which means that we must have

$$f = \frac{\partial b}{\partial u} \tag{11}$$

otherwise there exist no nontrivial conservation laws, whence we immediately get the first case of Theorem 1.

From now on we shall assume that (11) holds.

Outline of proof V

Using (11) one can simplify equation (9) to

$$\frac{\partial Q}{\partial t} + a\sum_{i,j=1}^{n}\frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + \frac{\partial g}{\partial u}Q + b\sum_{i=1}^{n}\frac{\partial^2 Q}{\partial x_i^2} = 0. \qquad (12)$$

Upon applying $\partial/\partial u$ to (12) we get

$$\frac{\partial^2 g}{\partial u^2}Q + \frac{\partial b}{\partial u}\sum_{i=1}^{n}\frac{\partial^2 Q}{\partial x_i^2} = 0. \qquad (13)$$

We can split the analysis of (13) into three cases which we label as A, B and C.
For the sake of brevity, let us state just the conditions and the results for these cases and then analyze one of them more in details as an example. Further details can be found in the article[8].

[8] P. Holba, Complete classification of local conservation laws for generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation. Stud Appl Math. https://doi.org/10.1111/sapm.12576

Case A: $\dfrac{\partial b}{\partial u} = 0$, meaning that b is constant and $f = 0$.

Setting $b = c_1$ and $g = c_2 u + c_3$, where c_1, c_2 and c_3 are arbitrary constants, we can simplify (12) to

$$\frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + c_2 Q + c_1 \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0. \tag{14}$$

Taking into account that we can assume without loss of generality that the associated conservation law densities depend at most on $x_1, \ldots, x_n, t$ and u, as discussed earlier, and making use of the formula $\partial T / \partial u = Q$ for the said densities we readily establish Theorem 2a.

Case B: $\dfrac{\partial b}{\partial u} \neq 0$ but $\dfrac{\partial^2 g}{\partial u^2} = 0$.

Then we get

$$Q = e^{-c_2 t}\tilde{Q},$$

where $\tilde{Q}$ is a (smooth) function of independent variables $x_1, \ldots, x_n$ which needs to satisfy the Laplace equation $\Delta(\tilde{Q}) = 0$, establishing Theorem 2b just like Theorem 2a above.

Case C: $\dfrac{\partial b}{\partial u} \neq 0$ and $\dfrac{\partial^2 g}{\partial u^2} \neq 0$.

Then we get

$$Q = e^{-(ac_4^2 - c_4 c_5)t}\tilde{Q},$$

where $\tilde{Q}$ is a (smooth) function of independent variables $x_1, \ldots, x_n$ which needs to satisfy $\tilde{Q} + \frac{1}{c_4}\Delta\tilde{Q} = 0$, thus establishing Theorem 2c in analogy with the previous case.

Outline of proof VIII

Let us examine *Case B* in a bit more detail as an example. We first recall condition (13) which reads

$$\frac{\partial^2 g}{\partial u^2} Q + \frac{\partial b}{\partial u} \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0.$$

The conditions $\partial b/\partial u \neq 0$ and $\partial^2 g/\partial u^2 = 0$ mean that we can set $g = c_2 u + c_3$ and then from (13) we get

$$\sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0 \tag{15}$$

Using these we can simplify condition (12) and get an easily solvable first-order equation,

$$\frac{\partial Q}{\partial t} + c_2 Q = 0. \tag{16}$$

whence $Q = e^{-c_2 t} \tilde{Q}$ with $\Delta(\tilde{Q}) = 0$ follows immediately. □

Conclusions

- For $n > 1$ the generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation

$$u_t = a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u)$$

has either infinitely many nontrivial local conservation laws or no nontrivial local conservation laws of any order.

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  has either infinitely many nontrivial local conservation laws or no nontrivial local conservation laws of any order.

- For $n = 1$ there are still infinitely many local conservation laws when the gCHKS equation is linear, but in the other cases there are just two local conservation laws.

Conclusions II

- In particular, the original Kuramoto–Sivashinsky equation

$$u_t + \Delta^2 u + \Delta u + |\nabla u|^2/2 = 0$$

has no nontrivial local conservation laws of any order for any space dimension $n = 1, 2, 3, \ldots,$

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$$u_t + \Delta^2 u + \Delta u + |\nabla u|^2/2 = 0$$

has no nontrivial local conservation laws of any order for any space dimension $n = 1, 2, 3, \dots,$

- The only nontrivial local conservation laws for the original Cahn–Hilliard equation

$$u_t = c_1 \Delta(u^3 - u + c_2 \Delta u)$$

with $c_1 \neq 0$ and $c_2 \neq 0$ are, modulo trivial ones, those with the densities of the form $Q(x_1, \dots, x_n)u$ where Q satisfies the Laplace equation $\Delta Q = 0$

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- For further details please see P. Holba, Complete classification of local conservation laws for generalized Cahn–Hilliard–Kuramoto–Sivashinsky equation. *Stud Appl Math.*, to appear, https://doi.org/10.1111/sapm.12576.

Thank you for your attention