Hamiltonian structure of rational isomonodromic deformation systems

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Outline

1. **Introduction: Isomonodromic deformations**
   - Isomonodromic systems history
   - Rational linear differential systems
   - Birkhoff deformation parameters
   - Jimbo-Miwa-Ueno isomonodromic deformation equations
2. **Hamiltonian structure: Rational R-matrix Poisson brackets**
   - Classical R-matrix theory
3. **Hamiltonian structure of isomonodromic deformations**
   - Birkhoff invariants as spectral invariants
   - Spectral invariant isomonodromic Hamiltonians
   - Isomonodromic τ-function
   - Hamiltonian vector fields and explicit derivatives
4. **Consistency conditions for explicit derivatives**
   - Explicit derivatives: consistency conditions, Poisson invariance
   - Transversality:Poisson quotient by abelian group action
   - Examples

Isomonodromic systems (history)

1900-1912: Isomonodromic deformations of linear differential systems with a finite number of isolated singular points : Painlevé , Fuchs, Garnier, Schlesinger and others (Picard, Gambier $\cdots$)

1913: Extension to generalized monodromy data including the Stokes and connection matrices in systems with irregular isolated singularities: Birkhoff

1980-81: A revival of interest in isomonodromic deformations was inspired by the theory of completely integrable systems: Flaschka and Newell and Jimbo, Miwa and Ueno

The presence of a *Hamiltonian structure*

was noticed since early studies of Painlevé equations (Fuchs (1905), Painlevé (1905), Malmquist (1922), Okamoto 1980), and isomonodromic τ-functions (Jimbo, Miwa, Ueno 1981). This was extended to more general systems using the classical rational R-matrix Poisson bracket on loop algebras (Harnad (1994), Boalch (2001)).

Rational covariant derivative operators:

Covariant derivative operator : $\mathcal{D}_z^L := \frac{\partial}{\partial z} - L(z), \quad z \in \mathbf{C},$

where $L(z)$ is a rational Lax matrix of the form

$$L(z) = -\sum_{j=0}^{d_\infty - 1} L_{j+2}^\infty z^j + \sum_{\nu=1}^{N} \sum_{j=1}^{d_\nu+1} \frac{L_j^\nu}{(z-c_\nu)^j} \in \mathcal{L}_{r,\mathbf{d}}$$

$$L_{d_\infty+1}^\infty \in \mathfrak{h}_{reg} \subset \mathfrak{gl}(r),\ L_{d_\nu+1}^\nu \in \mathfrak{g}_{reg} \subset \mathfrak{gl}(r), \quad c_\nu \neq c_\mu,\ \nu \neq \mu$$

$$r = \text{rank}, \quad d_\nu = \text{Poincaré index}, \quad \mathbf{d} := \{d_1, \dots, d_N, d_\infty\}.$$

with $L_{d_\infty+1}^\infty$ diagonal. Let $\Psi(z) \in \mathbf{Gl}(r, \mathbf{C})$ be a fundamental system of solutions to the linear system of first order ODE's

$$\frac{\partial \Psi(z)}{\partial z} = L(z)\Psi(z), \quad \Psi(z) \in \mathbf{Gl}(r, \mathbf{C}).$$

Theorem: Formal asymptotics and Birkhoff invariants

In terms of the local parameters

$$\zeta_\nu := (z - c_\nu), \quad \nu = 1, \dots, N, \quad \zeta_\infty := \frac{1}{z},$$

there exist local formal series solutions of the form

$$\Psi^\nu_{\text{form}}(z) = Y^\nu(\zeta_\nu) \mathrm{e}^{T^\nu(\zeta_\nu)}, \qquad Y^\nu(\zeta_\nu) := G^\nu \left(\mathbf{I} + \sum_{j \geq 1} Y^\nu_j {\zeta_\nu}^j \right),$$

in a punctured neighbourhood of each of the singular points $\{z = c_\nu\}$

Birkhoff invariants (contin.)

where $T^\nu(\zeta_\nu) \in \mathfrak{h}_{reg}$ is a diagonal $r \times r$ matrix of the form

$$T^\nu(\zeta_\nu) = \sum_{j=1}^{d_\nu} \frac{T_j^\nu}{j{\zeta_\nu}^j} + T_0^\nu \ln\zeta_\nu, \qquad T_{d_\nu}^\nu = -(G^\nu)^{-1} L_{d_\nu+1}^\nu G^\nu,$$

for $\nu = 1, \dots, N, \infty$. The columns of the invertible matrices $G^\nu \in GL(r, \mathbf{C})$ are the independent eigenvectors of $L_{d_\nu+1}^\nu$ and $G^\infty = \mathbf{I}$.

Notation for the diagonal values (Birkhoff invariants)

$$T_j^\nu = \mathrm{diag}(t_{j1}^\nu, \dots, t_{jr}^\nu), \quad j = 0, \dots, d_\nu$$

$$\text{so} \quad T^\nu(\zeta_\nu) = \sum_{j=1}^{d_\nu} \sum_{a=1}^{r} t_{ja}^\nu E_{aa} \frac{1}{j\zeta_\nu^j} + \sum_{a=1}^{r} t_{0a}^\nu E_{aa} \ln\zeta_\nu,$$

$$t_{ja}^\nu \neq t_{jb}^\nu \text{ for } a \neq b, \ \ j = 1, \dots, d_\nu, \ \ \nu = 1, \dots, N, \infty.$$

Definition: Infinitesimal isomonodromic deformation matrices

$$U^{\nu}_{ja}(z;L) := \left(Y^{\nu}(\zeta_\nu) \frac{\partial T^{\nu}(\zeta_\nu)}{\partial t^{\nu}_{ja}} (Y^{\nu}(\zeta_\nu))^{-1} \right)_{sing}$$
$$= \left(Y^{\nu}(\zeta_\nu) \frac{E_{aa}}{j\zeta_\nu^{j}} (Y^{\nu}(\zeta_\nu))^{-1} \right)_{sing},$$
$$V^{\nu}(z;L) := \left(Y^{\nu}(\zeta_\nu) \frac{\partial T^{\nu}(\zeta_\nu)}{\partial c_\nu} (Y^{\nu}(\zeta_\nu))^{-1} \right)_{sing}$$
$$= -\left(Y^{\nu}(\zeta_\nu) \frac{dT^{\nu}(\zeta_\nu)}{dz} (Y^{\nu}(\zeta_\nu))^{-1} \right)_{sing} = -\sum_{j=1}^{d_\nu+1} \frac{L^{\nu}_{j}}{(z-c_\nu)^{j}}.$$

where $(\cdot)_{sing}$ denotes the principal part of the Laurent series at a particular point $c_\nu \in \mathbf{P}^1$,

Theorem: Jimbo-Miwa-Ueno isomonodromic deformation equations (1981)

If the following system of (JMU) equations is satisfied

$$\frac{d\Psi(z)}{dz} = L(z)\Psi(z),$$
$$\frac{\partial\Psi(z)}{\partial t^{\nu}_{ja}} = U^{\nu}_{ja}(z)\Psi(z), \quad j = 1, \dots, d_{\nu},$$
$$\frac{\partial\Psi(z)}{\partial c_{\nu}} = V^{\nu}(z)\Psi(z), \quad \nu = 1, \dots, N,$$

the generalized monodromy (including the values of the Stokes matrices defined in a neighbourhood of each irregular singular point) is independent of the deformation parameters $\{t^{\nu}_{ja}, c_{\nu}\}$.
N.B. The exponents of formal monodromy $\{t^{\nu}_{0a}\}$, $\nu = 1, \dots, N, \infty$, do **not** occur as deformation parameters.

Consistency conditions: zero curvature equations

This overdetermined system is consistent if

$$\frac{\partial L(z)}{\partial t^{\nu}_{ja}} = \left[U^{\nu}_{ja}(z), L(z)\right] + \frac{dU^{\nu}_{ja}(z)}{dz},$$
$$\frac{\partial L(z)}{\partial c^{\nu}} = \left[V^{\nu}(z), L(z)\right] + \frac{dV^{\nu}(z)}{dz},$$
$$\frac{\partial U^{\mu}_{kb}(z)}{\partial t^{\nu}_{ja}} = \left[U^{\nu}_{ja}(z), U^{\mu}_{kb}(z)\right] + \frac{\partial U^{\nu}_{ja}(z)}{\partial t^{\mu}_{kb}},$$
$$\frac{\partial V^{\mu}(z)}{\partial c^{\nu}} = \left[V^{\nu}(z), V^{\nu}(z)\right] + \frac{\partial V^{\nu}(z)}{\partial c_{\nu}},$$

(etc.) is satisfied. (Zero curvature equations.)

Rational R-matrix Poisson brackets on phase space. (Also known as "linear Leningrad brackets".)

$$\{L_{ab}(z), L_{cd}(w)\} = \frac{1}{z-w}\Big((L_{ad}(z) - L_{ad}(w))\delta_{cb} - (L_{cb}(z) - L_{cb}(w))\delta_{ad}\Big).$$

Classical R-matrix theory then implies that:

- All elements of the ring $\mathcal{I}^{\mathrm{Ad}^*}(L^*\mathfrak{gl}(r))$ of Ad^* invariant functions of $L(z)$ (i.e., the ring of spectral invariants) Poisson commute.

$$\{f, g\} = 0, \quad \forall\, f, g \in \mathcal{I}^{\mathrm{Ad}^*}(L^*\mathfrak{gl}(r)).$$

$\mathcal{I}^{\mathrm{Ad}^*}(L^*\mathfrak{gl}(r))$ is generated by the coefficients of the characteristic polynomial defining the (planar) spectral curve.

$$\det(L(z) - \lambda \mathbf{I}) = 0.$$

Classical R-matrix theory (cont'd)

- The Hamiltonian vector field $\mathbf{X}_H$ generated by any element $H \in \mathcal{I}^{\mathrm{Ad}^*}(L^*\mathfrak{gl}(r))$ is given by a commutator

$$\mathbf{X}_H(X) = \{X, H\} = [R_s(dH), X], \qquad (2.1)$$
$$\forall\, H \in \mathcal{I}^{\mathrm{Ad}^*}(L^*\mathfrak{gl}(r)),\ X \in L\mathfrak{gl}(r),$$

where $X \in L\mathfrak{gl}(r)$ is viewed as a linear functional on $L^*\mathfrak{gl}(r)$ under the trace-residue pairing and R_s is the endomorphism of $L\mathfrak{gl}(r)$ defined by

$$R_s(Y_+ + Y_-) = sY_+ + (s-1)Y_-, \quad Y \in L\mathfrak{gl}(r)$$

for any $s \in \mathbb{C}$. In particular,

$$R_1(Y_+ + Y_-) = Y_+, \quad \text{and} \quad R_0(Y_+ + Y_-) = -Y_-.$$

Theorem (Birkhoff invariants as spectral invariants)

The matrix $\frac{dT^{\nu}(\zeta_\nu)}{d\zeta_\nu}$ *equals the principal part of the local Laurent series of the matrix* $\Lambda^{\nu}(\zeta_\nu) = \mathrm{diag}(\lambda_1^{\nu}, \cdots, \lambda_r^{\nu})$ *of eigenvalues near* $z = c_\nu$

$$\frac{dT^{\nu}}{d\zeta_\nu}(\zeta_\nu) = \left(\Lambda^{\nu}(\zeta_\nu)\right)_{sing},$$

where $\det(L(z) - \lambda_a^{\nu}\mathbf{I}) = 0$ *near* $z = c_\nu$,

$$\lambda_a^{\nu}(\zeta_\nu) = -\sum_{j=0}^{d_\nu} \frac{t_{ja}^{\nu}}{\zeta_\nu^{j+1}} + \mathcal{O}(1), \quad \nu = 1, \dots, N,$$

$$\lambda_a^{\infty}(\zeta_\infty) = \sum_{j=0}^{d_\infty} \frac{t_{ja}^{\infty}}{\zeta_\infty^{j-1}} + \mathcal{O}(\zeta_\infty^2),$$

and hence $t_{ja}^{\nu} = -\operatorname*{res}_{z=c_\nu} \zeta_\nu^{j} \lambda_a(z) dz,$

$$j = 1, \dots, d_\nu, \quad \nu = 0, \dots, N, \infty, \quad a = 1, \dots, r.$$

Theorem (Hamiltonians as *dual spectral invariants*)

$$\lambda_a^\nu(\zeta_\nu) = -\sum_{j=1}^{d_\nu} \frac{t_{ja}^\nu}{\zeta_\nu^{j+1}} - \frac{t_{0a}^\nu}{\zeta_\nu} - \sum_{j=1}^{d_\nu} jH_{t_{ja}^\nu}\zeta_\nu^{j-1} + \mathcal{O}(\zeta_\nu^{d_\nu}), \quad \nu = 1, \dots, N,$$

$$\lambda_a^\infty(\zeta_\infty) = \sum_{j=1}^{d_\nu} \frac{t_{ja}^\infty}{\zeta_\infty^{j-1}} + t_{0a}^\infty\zeta_\infty + \sum_{j=1}^{d_\infty} jH_{t_{ja}^\infty}\zeta_\infty^{j+1} + \mathcal{O}(\zeta_\infty^{d_\infty+2}).$$

where the Hamiltonians are (when evaluated on solution manifolds)

$$H_{t_{ja}^\nu} := -\frac{1}{j}\operatorname*{res}_{z=c_\nu} \frac{1}{\zeta_\nu^j}\lambda_a(z)dz = -\operatorname*{res}_{z=c_\nu} \operatorname{tr}\left((Y^\nu)^{-1}\frac{dY^\nu}{dz}\frac{\partial T^\nu}{\partial t_{ja}^\nu}\right) dz,$$

$$\nu = 1, \dots, N, \infty, \quad j = 1, \dots, d_\nu, \quad a = 1, \dots, r,$$

$$H_{c_\nu} := \frac{1}{2}\operatorname*{res}_{z=c_\nu} \operatorname{tr}\Big(L^2(z)\Big)dz = H_{c_\nu}, = -\operatorname*{res}_{z=c_\nu} \operatorname{tr}\left((Y^\nu)^{-1}\frac{dY^\nu}{dz}\frac{\partial T^\nu}{\partial c_\nu}\right) dz$$

Casimir invariants

The Birkhoff invariants $\{t^{\nu}_{ja}\}_{\nu=1,\dots,N,\infty,\, j=1,\dots,d_\nu,\, a=1,\dots,r}$, the exponents of formal monodromy $\{t^{\infty}_{0a}\}_{j=1,\dots,d_\infty, a=1,\dots,r}$ at the finite poles and the pole loci $\{c_\nu\}_{\mu=1,\dots,N}$ are all Casimir elements of the Poisson structure. They are functionally independent, and generate the full ring of Casimir invariants (center) of the Poisson algebra.

Differentials on the space of deformation parameters.

$$\mathrm{d}_\nu := dc_\nu \frac{\partial}{\partial c_\nu} + \sum_{j=1}^{d_\nu} \sum_{a=1}^{r} dt^{\nu}_{ja} \frac{\partial}{\partial t^{\nu}_{ja}}, \quad \mathrm{d}_\infty := \sum_{j=1}^{d_\infty} \sum_{a=1}^{r} dt^{\infty}_{ja} \frac{\partial}{\partial t^{\infty}_{ja}} .$$

The differential:

$$\omega_{IM} := -\sum_{\nu=1}^{N,\infty} \operatorname*{res}_{z=c_\nu} \left(\mathrm{tr}\left((Y^\nu(\zeta_\nu))^{-1} \partial_z Y^\nu(\zeta_\nu) \mathrm{d}_\nu T^\nu(\zeta_\nu) \right) d\zeta_\nu \right)$$

is closed when restricted to the solution manifold of the isomonodromic equations and hence locally exact.

Isomonodromic τ-function (JMU, 1981)

The isomonodromic τ-function τ_{IM} is locally defined, up to a parameter independent normalization, by

$$\omega_{IM} := \mathrm{d}\ln\tau_{IM} = \sum_{\nu=1}^{N} H_\nu dc_\nu + \sum_{\nu=1}^{N,\infty} \sum_{j=1}^{d_\nu} \sum_{a=1}^{r} H_{t^\nu_{ja}} dt^\nu_{ja}.$$

Globally, it is a section of a line bundle over the space $\mathbf{T} := \{t^\nu_{ja}, c_\nu\}$ of deformation parameters.

Theorem (Hamiltonian vector fields)

$$\mathbf{X}_{H_{t^\nu_{ja}}} L := \Big[U^\nu_{ja}, L\Big], \quad \mathbf{X}_{H_{c_\nu}} L := \Big[V^\nu, L\Big],$$

$$R_0(dH_{t^\nu_{ja}}) = U^\nu_{ja}(z; L) = -(dH_{t^\nu_{ja}})_-, \quad R_0(dc^\nu) = V^\nu(z; L) - (dH_{c_\nu})_-,$$
$$\nu = 1, \dots, N, \quad R_1(dH_{t^\infty_{ja}}) = U^\infty_{ja} = (dH_{t^\infty_{ja}})_+,$$

Definition of *explicit derivatives w.o. deformation parameters. (Isomonodromic condition). "Trivial flat connection".*

$$\nabla_{t^\nu_{ja}} L(z) := \frac{d}{dz} U^\nu_{ja}(z; L), \qquad \nabla_{c^\nu} L(z) := \frac{d}{dz} V^\nu(z; L)$$

Q: In what sense are

$$\nabla_{t^\nu_{ja}} = \frac{\partial^0}{\partial^0 t^\nu_{ja}}, \quad \nabla_{c_\nu} = \frac{\partial^0}{\partial^0 c_\nu}$$

"explicit derivatives", defining a "trivial flat connection"?

Adding:$\mathbf{X}_{H_{t^\nu_{ja}}} + \nabla_{t^\nu}$ and $\mathbf{X}_{H_{c_\nu}} + \nabla_{c_\nu}$ gives the *zero-curvature equations*

$$\frac{\partial L(z)}{\partial t^\nu_{ja}} = \left[U^\nu_{ja}, L\right] + \frac{dU^\nu_{ja}(z)}{dz}$$
$$\frac{\partial L(z)}{\partial c^\nu} = \left[V^\nu, L\right] + \frac{dV^\nu(z}{dz}.$$

These are the *Consistency conditions for the JMU equations:*

$$\frac{d\Psi(z)}{dz} = L(z)\Psi(z), \quad \frac{\partial\Psi(z)}{\partial t^\nu_{ja}} = U^\nu_{ja}(z)\Psi(z), \quad j = 1, \dots, d_\nu,$$
$$\frac{\partial\Psi(z)}{\partial c_\nu} = V^\nu(z)\Psi(z), \quad \nu = 1, \dots, N.$$

guaranteeing the invariance of the generalized monodromy (including the values of the Stokes matrices) under changes in the deformation parameters $\{t^\nu_{ja}, c_\nu\}$.

Theorem (Consistency conditions for explicit derivatives)

For all $\mu, \nu = 1, \dots, N$, and $\nu = \infty$, the explicit derivative vector fields $\{\nabla_{c^\mu}, \nabla_{t^\nu_{ja}}\}_{j=1,\dots d_\nu, a=1,\dots,r}$ all commute amongst themselves, generating a (locally) free abelian group action that is transversal to the symplectic foliation, with

$$\nabla_t(s) = 0 \quad \forall\, t, s \in \{t^\nu_{jb}, c_\nu\}$$

Theorem (Invariance of Poisson brackets under ∇_t's)

Let t denote any of the isomonodromic deformation parameters $t \in \mathbf{T} := \{t^\nu_{ja}, c_\nu\}$ and ∇_t be the corresponding explicit derivative vector field. Then

$$\nabla_t\{f, g\} = \{\nabla_t f, g\} + \{f, \nabla_t g\}.$$

In particular, if f, g are in the joint kernel of all the ∇_t's, their Poisson bracket $\{f, g\}$ is also.

Transversal distribution

Let

$$\mathcal{T} := \mathrm{Span}\{\nabla_t,\ t \in \{t^{\nu}_{ja}, t^{\infty}_{ja}, c_{\nu}\}\}.$$

Proposition (*Poisson quotient by abelian group action*)

$\mathcal{T}$ is an integrable distribution of constant, maximal rank

$$N + r\sum_{\nu=1}^{N} d_{\nu} + r d_{\infty},$$

transversal to the symplectic foliation and the canonical projection $\pi : \mathcal{L}_{r,\mathbf{d}} \to \mathcal{W} := \mathcal{L}_{r,\mathbf{d}}/\mathcal{T}$ is Poisson.

Example 1. Schlesinger equations

$$\frac{\partial \Psi(z)}{\partial z} = L^{\mathrm{Sch}}(z)\Psi(z), \quad L^{Sch}(z) := \sum_{\nu=1}^{N} \frac{L^\nu}{z - c_\nu}$$

$$\frac{\partial L^\mu}{\partial c_\nu} = \frac{[L^\mu, L^\nu]}{c_\mu - c_\nu}, \quad \forall\, \nu \neq \mu, \quad \frac{\partial L^\mu}{\partial c_\mu} = - \sum_{\nu=1,\ \mu\neq\nu}^{N} \frac{[L^\mu, L^\nu]}{c_\mu - c_\nu},$$

Hamiltonians, τ-function, Isomonodromic condition

$$H_\nu := \frac{1}{2} \operatorname*{res}_{z=c_\nu} \operatorname{tr} \left(L^{Sch}\right)^2 dz, \quad d\ln(\tau^{Sch}) = \sum_{\nu=1}^{N} H_\nu dc_\nu.$$

$$R_0(dH_\nu) = -(dH_\nu)_- = -\frac{L^\nu}{z - c_\nu},$$

$$\frac{\partial L^{\mathrm{Sch}}}{\partial c_\nu} = \frac{\partial \left(\frac{-L^\nu}{z-c_\nu}\right)}{\partial z} = \frac{L^\nu}{(z - c_\nu)^2}.$$

2. Fuchsian plus Double pole at $z = \infty$.

$$\frac{\partial \Psi(z)}{\partial z} = L^{\mathrm{B}}(z)\Psi(z), \quad \Psi(z) \in \mathfrak{Gl}(r),$$
$$L^{B}(z) = B + L^{Sch}(z), \quad B = \mathrm{diag}(t_1^{\infty}, \dots, t_r^{\infty})$$

Spectral curve and invariants

$$\det(L^{B}(z) - \lambda_a \mathbf{I}) = 0$$
$$t_a^{\infty} = -\operatorname*{res}_{z=\infty} z^{-j}\lambda_a(z)dz, \quad K_a := -\operatorname*{res}_{z=\infty} \frac{1}{z^{-}}\lambda_a(z)dz$$

The Hamiltonians $\{K_a\}_{a=1,\dots,r}$ satisfy the isomonodromic condition

$$\frac{\partial (dK_a)_+}{\partial z} = \frac{\partial^0 L^B}{\partial t_a^{\infty}} (= E_{aa}), \quad a = 1, \dots, r$$

3. Hamiltonian structure of Painlevé P_{II} equation: $N = 0,\ r = 2,\ d_\infty = 3$

P_{II} equation:

$$u'' = 2u^3 + tu + \alpha,$$

Linear system:

$$\frac{\partial \Psi(z)}{\partial z} = L^{P_{II}}(z)\Psi(z), \quad \frac{\partial \Psi(z)}{\partial t} = U(z)\Psi(z)$$

$$L^{P_{II}}(z) := z^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + z \begin{pmatrix} 0 & -2y_1 \\ x_2 & 0 \end{pmatrix} + \begin{pmatrix} x_2 y_1 + \frac{t}{2} & -2y_2 \\ x_1 & -x_2 y_1 - \frac{t}{2} \end{pmatrix}$$

$$U(z) := \frac{z}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & -2y_1 \\ x_2 & 0 \end{pmatrix},$$

$$t = \frac{1}{2} \operatorname*{res}_{z=0} z^{-3} \mathrm{tr}\left(((L^{P_{II}})^{P_{II}})^2(z) \right) dz = 2t_{11}^\infty,$$

(Birkhoff Casimir at $z = \infty$)

3. P_{II} (cont'd). Spectral invariants

$$\lambda = \pm\sqrt{-\det(L)} = \pm\left(z^2 + \frac{t}{2} - \frac{x_1 y_1 + x_2 y_2}{z} + \frac{H_{II}}{z^2} + \dots\right),$$

$$H_{II} = \frac{1}{4}\operatorname*{res}_{z=0} z^{-1}\operatorname{tr}(L^2(z)) - \frac{t^2}{8} = \frac{1}{2}\left(x_2^2 y_1^2 + t x_2 y_1 - 2 x_1 y_2\right)$$

Isomonodromic condition:

$$\frac{\partial^0 L^{P_{II}}}{\partial^0 t} = \nabla_t(L^{P_{II}}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\partial U}{\partial z}.$$

3. Hamiltonian structure of Painlevé P_{II} (cont''d).

Choosing new canonical coordinates:

$$u := \frac{x_1}{x_2}, \quad v := x_2 y_1, \quad w := \ln x_2, \quad a := x_1 y_1 + x_2 y_2,$$
$$\theta = y_1 dx_1 + y_2 dx_2 = vdu + adw.$$

Hamiltonian, τ-function and autonomous spectral invariant :

$$H_{II} = \frac{1}{2}v^2 + \frac{1}{2}(t + 2u^2)v - au, \quad d\ln(\tau) = H_{II}dt,$$
$$a = -\frac{1}{4}\operatorname*{res}_{z=0} z^{-2}\mathrm{tr}(L(z))^2, \quad (w = \text{ignorable coordinate})$$

Hamilton's equations:

$$\frac{du}{dt} = v + u^2 + \frac{t}{2}, \quad \frac{dv}{dt} = -2uv + a,$$
$$\frac{d^2u}{dt^2} = 2u^3 + tu + \alpha, \quad \alpha = a - 1/2.$$

4. Higher order elements of P_{II} hierarchy: $N = 0$, $r = 2$, $d_\infty = 4$

Linear system:

$$\frac{\partial \Psi(z)}{\partial z} = L^{P_{II,2}}(z)\Psi(z), \quad \frac{\partial \Psi(z)}{\partial t_1} = U_1(z)\Psi(z), \quad \frac{\partial \Psi(z)}{\partial t_2} = U_2(z)\Psi(z)$$

$$\begin{aligned} L^{P_{II,2}}(z) = & \left(z^3 + (t_2 - x_1\, y_2)\, z - x_1\, y_3 - x_3\, y_2 + t_1\right) \sigma_3 \\ & - \sqrt{2}\left(x_1\left(z^2 + \frac{t_2}{2}\right) + x_3\, z + x_2 - \frac{1}{4} y_2\, {x_1}^2\right)\sigma_+ \\ & - \sqrt{2}\left(y_2\left(z^2 + \frac{t_2}{2}\right) + y_3\, z + y_1 - \frac{1}{4} x_1\, {y_2}^2\right)\sigma_- . \end{aligned}$$

Deformation matrices:

$$U_1 = \begin{bmatrix} z & -\sqrt{2}x_1 \\ -\sqrt{2}y_2 & -z \end{bmatrix}, \quad U_2 = \frac{1}{2}\begin{bmatrix} -x_1 y_2 + z^2 & -\sqrt{2}\,(x_1 z + x_3) \\ -\sqrt{2}\,(y_2 z + y_3) & x_1 y_2 - z^2 \end{bmatrix}$$

4. Higher order elements 0f P_{II} hierarchy (cont'd)

Spectral invariants

$$\lambda = z^3 + t_2 z + t_1 + \frac{a}{z} + \frac{H_1}{2z^2} + \frac{H_2}{2z^3} + \mathcal{O}(z^{-4}),$$

Exponent of formal monodromy at $z = \infty$

$$t_0^\infty := -\operatorname*{res}_{z=\infty} \sqrt{-\det L(z)}dz = a := x_1 y_1 + x_2 y_2 + x_3 y_3,$$

Canonical change of coordinates. Canonical 1-form.

$$\begin{aligned} &x_1 := u_1 e^w, \quad x_2 := u_2 e^w, \quad x_3 := e^w, \\ &y_1 := v_1 e^{-w}, \quad y_2 := v_2 e^{-w}, \quad y_3 := (a - u_1 v_1 - u_2 v_2) e^{-w}, \\ &\theta = \sum_{i=1}^3 y_i dx_i = v_1 du_1 + v_2 du_2 + a dw, \end{aligned}$$

4. Higher order elements 0f P_{II} hierarchy (cont'd)

The reduced Hamiltonians are then:

$$
\begin{aligned}
H_1 = &\left(\frac{3}{2}v_2{u_1}^2 - t_2 u_1 + 2u_2\right) a - 2t_1 u_1 v_2 + \left({u_1}^2 v_1 + u_1 u_2 v_2 - v_2\right) t_2 \\
&- \frac{3}{2}{u_1}^3 v_1 v_2 - \frac{3}{2}{u_1}^2 u_2 {v_2}^2 - 2u_1 u_2 v_1 + \frac{3}{2}u_1 {v_2}^2 - 2{u_2}^2 v_2 + 2v_1, \\
H_2 = &\frac{1}{2}a^2{u_1}^2 + \left(-u_1 t_1 - t_2 - u_1\left({u_1}^2 v_1 + u_1 u_2 v_2 - v_2\right)\right) a \\
&+ \left({u_1}^2 v_1 + u_1 u_2 v_2 - v_2\right) t_1 + \frac{1}{4}{t_2}^2 u_1 v_2 \\
&+ \left(-\frac{1}{4}{v_2}^2{u_1}^2 + \frac{1}{2}u_1 v_1 + \frac{1}{2}u_2 v_2\right) t_2 + \frac{1}{2}{u_1}^4{v_1}^2 + {u_1}^3 u_2 v_1 v_2 \\
&+ \frac{1}{16}{v_2}^3{u_1}^3 + \frac{1}{2}{u_1}^2{u_2}^2{v_2}^2 - \frac{5}{4}{u_1}^2 v_1 v_2 - \frac{5}{4}u_1 u_2 {v_2}^2 + \frac{1}{2}{v_2}^2 + u_2 v_1.
\end{aligned}
$$

where w is a completely ignorable canonical coordinate conjugate to the conserved spectral invariant a.

Further developments: Darboux coordinates

To express all higher isomonodromic deformation equations in explicitly Hamiltonian form, we would need, in addition to the Casimir invariant coordinate functions $\{t^{\nu}_{ja}, c_{\nu}\}$, a set of Darboux coordinates $\{u_\alpha, v_\alpha\}_{\alpha=1,\dots,K}$ on the symplectic leaves that are invariant under the integrable distribution $\mathcal{T}$ corresponding to the trivial (flat) connection ∇ defining the explicit derivatives of L.

$$\nabla_t u_\alpha = 0, \quad \nabla_t v_\alpha = 0, \quad \forall\, \alpha = 1, \dots, K$$
$$2K := r(r-1)\left(d_\infty + \sum_{\nu=1}^{N} d_\nu + N - 1\right).$$

An attempt in this direction was made by Marchal, Orantin and Alalameddine (2022) for rank $r = 2$, using spectral Darboux coordinates, a different trivialization of the bundle, and different choices of Hamiltonians. To relate the two, a multi-time dependent canonical transformation is required.

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