Reflection Equation Algebra and related combinatorics

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Let V be a vector space over the field $\mathbb{C}$. We say that a linear invertible operator $R: V^{\otimes 2} \to V^{\otimes 2}$ is a braiding, if it is subject to the following relation in $V^{\otimes 3}$

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R),$$

where $I: V \to V$ is the identity operator.

The simplest example is the usual flip $R = P$, which acts as follows $P(x \otimes y) = y \otimes x$ for any $x, y \in V$, or a super-flip denoted $P_{m|n}$. This notation means that $dim V_0 = m$ and $dim V_1 = n$, where we assume V to be a super-space $V = V_0 \oplus V_1$. The component V_0 is even and that V_1 is odd. The parity of an element $x \in V$ is denoted $\overline{x}$. So, the super-flip is as follows

$$P_{m|n}(x \otimes y) = (-1)^{\overline{x}\,\overline{y}} y \otimes x.$$

Note that $P^2_{m|n} = I$. Also, note that the usual flip P is a particular case of a super-flip $n = 0$. The braidings R subject to the condition $R^2 = 1$ are called involutive symmetries. The braidings subject to the Hecke condition

$$(q\, I - R)(q^{-1}\, I + R) = 0,\;\; q \in \mathbb{C},\;\; q \neq 0,\;\; q \neq \pm 1$$

are called Hecke symmetries.
For any braiding R we denote $R_k : V^{\otimes p} \to V^{\otimes p},\;\; k = 1, ...p - 1$ the operator R acting on the components numbers k and $k + 1$. Thus,

$$R_k = I_{1...k-1} \otimes R \otimes I_{k+2...p}.$$

Observe that the braidings and Hecke symmetries are in fact representations of the braid groups and Hecke algebras. We call these representations R-matrix ones.
Recall that The Artin braid group B_N is the group generated by the unit e and $N-1$ invertible elements

$$\tau_1, \ldots, \tau_{N-1}$$

subject to the following relations

$$\tau_i\,\tau_j = \tau_j\,\tau_i \text{ if } |i-j| \geq 2 \text{ and } \tau_i\,\tau_{i+1}\,\tau_i = \tau_{i+1}\,\tau_i\,\tau_{i+1},\ \ i \leq N-2.$$

The last relation is called braid one.
If we impose the following relation

$$(\tau_i - q\,e)(\tau_i + q^{-1}\,e) = 0,\ \forall i,\ \ q \in \mathbb{C},$$

we get an algebra $H_N(q)$ called Hecke (or Iwahori-Hecke) algebra.
Observe that for $q = \pm 1$ we get the algebra $\mathbb{C}[\mathbb{S}_N]$.
We deal with q generic: $q \notin \{0, \pm 1\}$ and q is not a root of unity.

By fixing in the space V a basis $\{x_1 \dots x_N\}$ and the corresponding basis $\{x_i \otimes x_j\}$ in the space $V^{\otimes 2}$ we can represent the operators R_i by matrices. Let us exhibit two examples of Hecke symmetries

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

The first matrix tends to the matrix of the flip P as $q \to 1$. The second one tends to the super-flip $P_{1|1}$.
Note that if R is a braiding, then the operator $\mathcal{R} = R\,P$, where P is the usual flip, meets the so-called Quantum Yang-Baxter equation

$$\mathcal{R}_{12}\,\mathcal{R}_{13}\,\mathcal{R}_{23} = \mathcal{R}_{23}\,\mathcal{R}_{13}\,\mathcal{R}_{12}.$$

Note that there exist involutive and Hecke symmetries which are deformations neither of the usual flips nor of the super-flips. So, the following problem is of interest; what involutive and Hecke symmetries could be. Consider two related algebras: "R-symmetric" and "R-skew-symmetric" ones

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V) = \bigoplus V^{\otimes k}$ is the free tensor algebra of V.
Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V)t^k, \ P_-(t) = \sum_k dim \bigwedge_R^{(k)}(V)t^k,$$

where the upper index (k) labels the homogenous components.
If R is involutive, we put $q = 1$ in the above formulae.

Examples. If R is a deformation of the usual flip P and $dim V = N$, then

$$P_-(t) = (1+t)^N.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exit involutive and Hecke symmetries such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dim V = N$.
If $P_-(t)$ is a polynomial, R is called *even*.

Proposition. (G)

For a generic q the following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^{r}(1 + x_i t)}{\prod_{j=1}^{s}(1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all involutive and Hecke symmetries are similar to super-flips, for which the role of the bi-rank is played by the super-dimension $(m|n)$.

Note that all numerical characteristic of the related objects are expressed via the bi-rank $(r|s)$ of the initial involutive or Hecke symmetry R.

The bi-rank enters the quantum dimension of V and other spaces and other numerical characteristics.

Example.

The usual dimension of a super-space $V = V_0 \oplus V_1$ with super-dimension $(m|n)$ is $N = m + n$, whereas its quantum dimension is $m - n$.

Now consider other algebras, associated to involutive or Hecke symmetries. Let R be such a symmetry. And let $L = \|l_i^j\|$ be a $N \times N$ matrix. Consider the relation

$$RL_1RL_1 - L_1RL_1R = 0, \; L = (l_i^j), \; 1 \le i, j \le N,$$

where $L_1 = L \otimes I$. The algebra generated by the unity and the entries l_i^j, subject to the above relation, is called Reflection Equation (RE) one and denoted $\mathcal{L}(R)$.
If a $N \times N$ matrix $\hat{L} = \|\hat{l}_i^j\|$ is subject to

$$R\hat{L}_1R\hat{L}_1 - \hat{L}_1R\hat{L}_1R = R\hat{L}_1 - \hat{L}_1R, \; \hat{L} = (\hat{l}_i^j), \; 1 \le i, j \le N,$$

the algebra generated by the unity and the entries $\hat{l}_i^j$ is called modified RE algebra and denoted $\hat{\mathcal{L}}(R)$.
Nevertheless, these algebras are isomorph to each other. Their isomorphism can be realised via the following relations between the generating matrices

$$L = I - (q - q^{-1})\,\hat{L}.$$

Observe that this isomorphism fails if $q = \pm 1$.

Nevertheless, if $R \to P$ as $q \to 1$, the RE algebras $\mathcal{L}(R)$ tends to the algebra $Sym(gl(N))$, whereas the modified RE algebra $\hat{\mathcal{L}}(R)$ tends to that $U(gl(N)))$.
Note that the algebra $Sym(gl(N))$ can be presented in the matrix form

$$PL_1PL_1 - L_1PL_1P = 0, \;\; L = (l_i^j), \;\; 1 \le i,j \le N,$$

whereas the algebra $U(gl(N)))$ can be presented by the system

$$P\hat{L}_1P\hat{L}_1 - \hat{L}_1P\hat{L}_1P = P\hat{L}_1 - \hat{L}_1P, \;\; \hat{L} = (\hat{l}_i^j), \;\; 1 \le i,j \le N,$$

Our next aim is to describe the center of the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ and to introduce analogs of the symmetric polynomials on them.

Observe that the center of the algebra $U(gl(N))$ is generated by the elements $Tr\hat{L}^k$, which are called power sums.
In a similar manner we introduce "power sums" in the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ but the trace must be "quantum".
Let us introduce the so-called R-trace of matrices

$$Tr_R M = Tr C\, M,$$

where the matrix $C = (C_i^j)$ is completely defined by a given braiding R (it can be defined for all braidings, which are "skew-invertible").
Then the power sums in the algebras $\mathcal{L}(R)$ and $\hat{\mathcal{L}}(R)$ are defined as follows

$$p_k(L) = Tr_R\, L^k = Tr\, C\, L^k, \;\; p_k(\hat{L}) = Tr_R\, \hat{L}^k = Tr\, C\, \hat{L}^k.$$

These elements are central in the corresponding algebras.
This is one of the main difference between the RE algebras (modified or not) and the so-called RTT algebras where similar elements are not central but generate a commutative subalgebras (called Bethe).

Now, we want to describe other elements of the center of the algebra $\mathcal{L}(R)$.
Below, we use the following notations

$$L_{\overline{1}} = L_1, \; L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1}, \; L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \ldots$$

Also, we use the notation

$$L_{\overline{1\to k}} = L_{\overline{1}}\, L_{\overline{2}} \ldots L_{\overline{k}}.$$

Example. If $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then

$$L_{\overline{1}} = L_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & b & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \; L_{\overline{2}} = R \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix} R^{-1}.$$

The following claim can be found in the paper by Isaev-Pyatov.

Proposition.

Let $z \in H_k(q)$ be an arbitrary element. Then the element

$$ch(z) := Tr_{R(1\dots k)}\, \rho_R(z) L_{\overline{1\to k}}$$

is central in the algebra $\mathcal{L}(R)$.

Here $\rho_R(z)$ is R-matrix representation of the Hecke algebra

$$\rho_R(\tau_k) = R_k.$$

The map

$$ch : H_k(q) \to \mathcal{L}(R), \;\; z \mapsto ch(z)$$

is called characteristic.

Observe that the power sums $p_k(L)$ in the algebra $\mathcal{L}(R)$ can be written in this form.
Also, in this manner we introduce the Schur polynomials (functions) in this algebra $\mathcal{L}(R)$. In order to introduce them we put $z = E^{\lambda}_{ii}$, where E^{λ}_{ii} some idempotents belonging to the Hecke algebra. Thus, we put

$$s_\lambda(L) = Tr_{R(1\dots k)}\, \rho_R(z) L_{\overline{1\to k}}, \quad z = E^{\lambda}_{ii}.$$

Now, we exhibit the classical Frobenius formula and its q-analog.

The classical Frobenius formula is

$$p_\nu(L) = \sum_{\lambda \vdash k} \chi_\nu^\lambda s_\lambda(L),$$

where $\nu = (\nu_1, \nu_2 ... \nu_k)$ and $\lambda = (\lambda_1 ... \lambda_k)$ are partitions, $p_\nu(L) = p_{\nu_1} ... p_{\nu_k}$ are the corresponding power sums, and $s_\lambda(L)$ are the Schur polynomials. (Here L is the generating matrix of the algebra $Sym(gl(N))$.) Also, χ_ν^λ is is the character of the symmetric groups in the representation, labeled by λ, evaluated on the element whose cyclic type is $\nu = (\nu_1 ... \nu_N)$.
The q-analog of the Frobenius formula looks like the classical one but L becomes the generating matrix of the algebra $\mathcal{L}(R)$ and χ_ν^λ is the character of the representation of the Hecke algebra.

Note that a similar formula has been obtained by Arum Ram but only in the case related to the QG $U_q(sl(N))$. In general the main problem is how to correctly define the symmetric function related to RE algebras.

Note that our definition of all symmetric functions in question differs from the classical one.
Now, we want to represent them under a form more similar to the classical one. In order to do so we introduce "eigenvalues" of the matrices L generating the RE algebras $\mathcal{L}(R)$.

Observe that in the algebra $\mathcal{L}(R)$ there are analogs of the Newton identities

$$p_k - q p_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \ldots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0,$$

$k = 1.2\ldots$ and the Cayley-Hamilton identity

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0,$$

provided R is even of bi-rank $(m|0)$.
Here, $e_k = e_k(L)$ are elementary symmetric polynomials (functions), i.e. particular cases of the Schur polynomials, respective to one-column diagrams. Recall that all these elements are central in the RE algebra.

Let $\mu_i,\ i = 1...m$ be indeterminates meeting the following system

$$\sum_i \mu_i = e_q(L),\ \sum_{i<j} \mu_i\, \mu_j = q^2\, e_2(L),\ \dots, \mu_1\, \mu_2 ... \mu_m = q^m\, e_m(L).$$

We call μ_i (quantum) "eigenvalues" of the matrix L. These indeterminates are assumed to be central in the algebra $\mathcal{L}(R)[\mu_1 ... \mu_m]$.

If R is not even and its bi-rank is $(m|n)$, then the eigenvalues can be defined in a similar way. In this case we have two families of them $\mu_1, ..., \mu_m$ (even eigenvalues) and $\nu_1, ..., \nu_n$ (odd eigenvalues) such that any symmetric polynomial can be expressed via these quantities.

For the power sums we have the following parametrization via the "eigenvalues" μ_i and ν_i:

$$p_k(L) = Tr_R\, L^k = \sum_i^m \mu_i^k\, d_i + \sum_j^n \nu_j^k\, \tilde{d}_j,$$

$$d_i = q^{-1} \prod_{p=1, p\neq i}^m \frac{\mu_i - q^{-2}\mu_p}{\mu_i - \mu_p} \prod_{j=1}^n \frac{\mu_i - q^2\nu_j}{\mu_i - \nu_j},$$

$$\tilde{d}_j = -q \prod_{i=1}^m \frac{\nu_j - q^{-2}\mu_i}{\nu_j - \mu_i} \prod_{p=1,\, p\neq j}^n \frac{\nu_j - q^2\nu_p}{\nu_j - \nu_p},$$

In the limit $q = 1$ we get the formula, corresponding to the involutive symmetry R

$$p_k(L) = \sum_i^m \mu_i^k - \sum_j^n \nu_j^k.$$

In the even case these polynomials coincide with the Hall-Littlewood polynomials up to numerical factors q and q^{-1} and identification $t = q^{-2}$.

Observe that these polynomials are super-symmetric in $q^{-1}\,\mu_i$ and $q\,\nu_j$.

Recall that by definition a polynomial in two sets in indeterminates μ_i and ν_j is called super-symmetric if it is symmetric in μ_i and ν_j separately and the polynomial in which one puts $\mu_1 = \nu_1 = s$ does not depend on s.

Note that in the even case (i.e. while R is even), the elementary symmetric functions are usual elementary symmetric functions in $q^{-1}\,\mu_i$ by definition.

Besides, the classical Littlewood-Richardson rule is still valid. So, other Schur functions (including full elementary functions) can be expressed via the elementary symmetric functions in $q^{-1}\,\mu_i$.

Thus, the only essential difference with the usual calculus is that the role of power sums on the RE algebras, expressed in the variables $q^{-1}\,\mu_i$, is played by the Hall-Littlewood polynomials (up to a factor and the substitution $t = q^{-2}$).

The second application of the RE algebras is a q-analog of the Capelli formula. First, recall this formula in the classical setting. Again, consider a matrix L with commuting entries l_i^j and the matrix D composed from the partial derivatives ∂_i^j such that

$$\partial_i^j(l_k^l) = \delta_i^l \delta_k^j.$$

Then the matrix

$$\hat{L} = L\,D$$

generates the enveloping algebra $U(gl(N))$.
The classical Capelli identity is

$$rDet(\hat{L} + K) = detL\, detD,$$

where K is the diagonal matrix $diag(0, 1, ..., n-1)$ and $rDet$ is the so-called row-determinant.

By passing to the q-case we replace the above commutative algebra by the RE one, generated by the entries of the matrix $L = \|l_i^j\|$. We succeeded to introduce "quantum partial derivatives" in the generators l_i^j of this algebra $\partial_i^j = \partial_{l_j^i}$. Let D be the matrix composed from this derivatives. The following holds

Theorem.

Let $L = \|l_i^j\|_{1\le i,j\le N}$ be the generating matrix of an algebra $\mathcal{L}(R)$ and $D = \|\partial_i^j\|_{1\le i,j\le N}$ be the matrix composed from the partial derivatives. Then the matrix

$$\hat{L} = L\,D$$

generates the modified RE algebra.

Proposition.

In the RE algebra the following holds

$$Tr_{R(1\dots m)}\, A^{(m)}\, \hat{L}_1\, (\hat{L}_{\overline{2}} + q\, I)(\hat{L}_{\overline{3}} + q^2\, 2_q\, I)\dots(\hat{L}_{\overline{m}} + q^{m-1}\, (m-1)_q\, I) =$$

$$q^{-m}\, det_R\, L\, det_{R^{-1}}\, D.$$

Here $(m|0)$ is the bi-rank of R and $A^{(m)}$ is the highest R-skew-symmetrizer

Also, the determinants are the highest elementary polynomials, which can be defined for any even symmetry.