CONTACT GEOMETRY AS A CHAPTER IN SYMPLECTIC GEOMETRY

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IN MEMORY OF MY FRIEND

ANATOL ODZIJEWICZ

Contact structures

- $C \subset \mathsf{T}M$ – a corank-1 distribution on M
- $\tau_C : \mathsf{T}M \to L_C = \mathsf{T}M/C$ – the canonical projection.

Definition

A hyperplane field $C \subset \mathsf{T}M$ we call a contact structure if the 2-form on C,

$$\nu_C : C \times_M C \to L_C,\ \nu_C(X,Y) = \tau_C([X,Y]),$$

is nondegenerate. Any nonvanishing (local) 1-form η such that $C = \ker(\eta)$ we call a contact form generating C.

- $\dim(M) = 2n+1$
- Characterization of contact forms: $\eta \wedge (\mathrm{d}\eta)^n \neq 0$.
- $\ker(\eta) = \ker(f\eta)$ if $f \neq 0$ – conformal equivalence

Darboux Theorem and Reeb vector field

- A contact manifold (M, C) we call trivial (or co-oriented) if there is fixed a global contact form η such that $C = \ker(\eta)$.
  (M, C) is trivializable (co-orientable) if there exists a global contact form η such that $C = \ker(\eta)$.
- Important: Not all contact manifolds are trivializable. Examples will follow.

On every trivial contact manifold (M, η) there is a unique vector field $\mathcal{R}$ (the Reeb vector field) such that

$$\langle \eta, \mathcal{R} \rangle = 1 \text{ and } i_{\mathcal{R}} \mathrm{d}\eta = 0.$$

Theorem (Contact Darboux Theorem)

If η is a contact form, then locally $\eta = \mathrm{d}z - p_i\, \mathrm{d}q^i$.

$\mathrm{d}\eta = \mathrm{d}p_i \wedge \mathrm{d}q^i$ and $\mathcal{R} = \partial_z$. Similar η lives on $\mathbb{R}^* \times \mathsf{T}^*Q$.

Contact dynamics

In physics literature: thermodynamics (equilibrium and non-equilibrium), statistical mechanics, Hamiltonian and Lagrangian mechanics, study of systems with dissipation, etc., contact structures are generally co-oriented: (M, η).

The contact dynamics $X^c_{\hat{H}}$ associated with a 'contact Hamiltonian' $\hat{H} : M \to \mathbb{R}$ (*ad hoc* definition):

$$i_{X^c_{\hat{H}}}\eta = -\hat{H}, \qquad i_{X^c_{\hat{H}}}\mathrm{d}\eta = \mathrm{d}\hat{H} - \mathcal{R}(\hat{H})\eta.$$

Strongly depends on the choice of η even on trivializable contact manifolds.

In Darboux coordinates,

$$X^c_{\hat{H}} = \frac{\partial \hat{H}}{\partial p_i}\partial_{q^i} - \left(\frac{\partial \hat{H}}{\partial q^i} + \frac{\partial \hat{H}}{\partial z}p_i\right)\partial_{p_i} + \left(p_i\frac{\partial \hat{H}}{\partial p_i} - \hat{H}\right)\partial_z.$$

Contactomorphisms

Definition

Contactomorphisms between contact manifolds (M_1, C_1) and (M_2, C_2) are diffeomorphisms $\varphi : M_1 \to M_2$ such that $\mathsf{T}\varphi(C_1) = C_2$.

Contactomorphisms map (local) contact forms determining the contact distributions into (local) contact forms in the same class of conformal equivalence,

$$\varphi^*(\eta_2) = f\eta_1,\ f \neq 0.$$

Contact vector fields on a contact manifold (M, C) are vector fields on M whose (local) flows are (local) contactomorphisms:

$$\pounds_X(Y) = [Y, X] \in C \text{ if } Y \in C.$$

Double vector bundles

Theorem (Grabowski-Rotkiewicz)

Every vector bundle $\tau : E \to M$ is uniquely determined by the multiplication by reals on E, i.e., by homotheties

$$h_s : E \to E,\ h_s(v) = sv$$

(equivalently, by the Euler vector field ∇_E). Note that $\tau = h_0$. A double vector bundle (DVB) is a manifold F with two VB structures whose homotheties commute,

$$h^1_t \circ h^2_s = h^2_s \circ h^1_t,$$

(whose Euler vector fields commute, $[\nabla_1, \nabla_2] = 0$).

Example

$$\begin{array}{ccc} \mathsf{T}E & \xrightarrow{\mathsf{T}\tau} & \mathsf{T}M \\ {\scriptstyle \tau_E}\downarrow & & \downarrow{\scriptstyle \tau} \\ E & \xrightarrow{\tau} & M \end{array} \qquad \begin{array}{ccc} \mathsf{T}^*E & \xrightarrow{\mathsf{T}^*\tau} & E^* \\ {\scriptstyle \pi_E}\downarrow & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\tau} & M \end{array}$$

Digressions on linearity

Additional multiplications by reals (homotheties):

$$(\mathrm{d}_{\mathsf{T}}h)_s = \mathsf{T}h_s, \quad (\mathrm{d}_{\mathsf{T}^*}h)_s = s(\mathsf{T}h_{s^{-1}})^*.$$

$$\boxed{\text{DVB} \rightsquigarrow \text{linearity}}$$

- a vector field X on E is linear $\Leftrightarrow X : E \to \mathsf{T}E$ is a morphism of VB: $\tau : E \to M$ into $\mathsf{T}\tau : \mathsf{T}E \to \mathsf{T}M$;
- a 2-form ω on E is linear $\Leftrightarrow h_s^*(\omega) = s\omega$ $\Leftrightarrow \omega^\flat : \mathsf{T}E \to \mathsf{T}^*E$ is a morphism of DVB;
- A distribution C on E is linear $\Leftrightarrow \mathsf{T}h_s(C) \subset C$ $\Leftrightarrow C$ is a double vector subbundle of $\mathsf{T}E$

Cotangent bundles are the only linear symplectic manifolds.

$\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ – the multiplicative group of invertible reals. There is an analogy between VB and $\mathbb{R}^\times$-principal bundles $\tau : P \to M$: the principal action $h_s : P \to P$ is defined only for $s \neq 0$ and linearity is replaced by homogeneity.

Vector and $\mathbb{R}^\times$-principal bundles

For a vector bundle $\tau : E \to M$ denote
$$E^\times = \{v \in E : v \neq 0\} = E \setminus 0_M.$$

Note that $E^\times$ is always an $\mathbb{R}^\times$-principal bundle with the action by homotheties h_s restricted to $s \neq 0$.

Line (rank 1) bundles $L \leftrightarrow$ Principal $\mathbb{R}^\times$-bundles $P = L^\times$

$L = P \times_{\mathbb{R}^\times} \mathbb{R}$ – the associated line bundle.

Lifts of an $\mathbb{R}^\times$-principal action on P to $\mathsf{T}P$ and T^*P like for VB: $(\mathrm{d}_{\mathsf{T}} h)_s = \mathsf{T}h_s, \quad (\mathrm{d}_{\mathsf{T}^*} h)_s = s(\mathsf{T}h_{s^{-1}})^*,\ s \neq 0$.

Note that $\mathrm{d}_{\mathsf{T}^*} h$ is not the standard lift of a group action to T^*P which is $(\mathsf{T}h_{s^{-1}})^*$.

Double structures, $\mathbb{R}^\times$-principal + VB structures are clear: the $\mathbb{R}^\times$- and $\mathbb{R}$-actions commute, e.g., $\mathsf{T}P,\ \mathsf{T}^*P$.

Symplectizations of contact forms

Let η be a nonvanishing 1-form on M. Then, η spans a line subbundle $[\eta]$ in the vector bundle T^*M. Hence, $[\eta]^\times$ is canonically an $\mathbb{R}^\times$-principal bundle with the $\mathbb{R}^\times$-action inherited from the multiplication by reals in T^*M.
With ω_M we denote the canonical symplectic form on T^*M.

Proposition

A nonvanishing 1-form η on M is a contact form if and only if the $\mathbb{R}^\times$-principal bundle $[\eta]^\times \subset \mathsf{T}^*M$ is a symplectic submanifold of $(\mathsf{T}^*M, \omega_M)$.

$$\Phi_\eta : \mathbb{R}^\times \times M \ni (s,x) \mapsto s\eta(x) \in [\eta]^\times \subset \mathsf{T}^*M$$

is an isomorphism of $\mathbb{R}^\times$-bundles.

$$\omega_\eta = \Phi_\eta^*(\omega_M)(s,x) = \mathrm{d}(s\eta)(s,x) = \mathrm{d}s \wedge \eta(x) + s\,\mathrm{d}\eta(x)\,.$$

Hence, $\boxed{\eta \text{ is a contact form if and only if } \omega_\eta \text{ is symplectic}}$.

Symplectizations of contact manifolds

Note that $[\eta] = (\ker(\eta))^o$ depends only on the conformal class of η.

Corollary

A corank-1 distribution $C \subset \mathsf{T}M$ is a contact structure if and only if $P = (C^o)^\times = (L^*_C)^\times$ is a symplectic submanifold in T^*M, with the symplectic form $\omega_C = \omega_M\big|_P$.
If η is a local contact form determining C on $U \subset M$, then $P\big|_U = \mathbb{R}^\times \times U$ and

$$\omega_C(s,x) = \mathrm{d}s \wedge \eta(x) + s\,\mathrm{d}\eta(x).$$

Attention: The $\mathbb{R}^\times$-bundle $P = (C^o)^\times$ is generally not trivializable.
Since ω_M is linear, the symplectic form ω_C on the $\mathbb{R}^\times$-principal bundle $P = (C^o)^\times$ is homogeneous: $h^*_s(\omega_C) = s\omega_C$, $s \neq 0$.

Symplectic $\mathbb{R}^\times$-bundles

This can be expressed also by: $\pounds_\nabla \omega_M = \omega_M$ on T^*M, and $\pounds_{\nabla_P} \omega_C = \omega_C$, where ∇_P is the restriction of the Euler vector field ∇ to P – the generator of the $\mathbb{R}^\times$-action h_s.

Definition

A symplectic $\mathbb{R}^\times$-bundle (P, τ, M, h, ω) is a principal $\mathbb{R}^\times$-bundle $\tau : P \to M$ with an $\mathbb{R}^\times$-action

$$h : \mathbb{R}^\times \times P \to P\,, \quad \mathbb{R}^\times \times P \ni (s, x) \mapsto h_s(x) \in P\,,$$

equipped additionally with a symplectic form ω which is 1-homogeneous, $(h_s)^*(\omega) = s \cdot \omega$.

Isomorphisms of symplectic $\mathbb{R}^\times$-bundles are $\mathbb{R}^\times$-equivariant symplectomorphisms.

Euler vector field and Liouville 1-form

The following objects live on every symplectic $\mathbb{R}^\times$-bundle:

- the Euler vector field, being the generator of the $(\mathbb{R}_+,\cdot)$-action,
$$\nabla_P(x) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(h_{e^t}(x)) = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=1}(h_t(x)),$$
- and a nonvanishing 1-form $\theta = i_{\nabla_P}\omega$, the Liouville form.

Theorem

If (P,τ,M,h,ω) is a symplectic $\mathbb{R}^\times$-bundle, then θ is semi-basic, and $\omega = \mathrm{d}\theta$. For any local trivialization $P\big|_U \simeq \mathbb{R}^\times \times U$ of the principal bundle $\tau : P \to M$, we have $\nabla_P = s\partial_s$, and there is a contact form η on U such that
$$\omega(s,x) = \mathrm{d}s \wedge \eta(x) + s\cdot \mathrm{d}\eta(x);$$
$$\theta(s,x) = s\cdot\eta(x).$$

Contact mfd = symplectic $\mathbb{R}^\times$-bundle

Theorem

Any symplectic $\mathbb{R}^\times$*-bundle* (P, τ, M, h, ω) *induces canonically a contact structure* $C = \mathsf{T}\tau(\ker(\theta))$ *on* M*, together with an isomorphism*

$$\begin{array}{ccc} P & \xrightarrow{\Phi_P} & (C^o)^\times \\ \tau \downarrow & & \downarrow \pi_M \\ M & \xrightarrow{\mathrm{id}_M} & M \end{array}$$

of symplectic $\mathbb{R}^\times$*-bundles. Here,* θ *is the Liouville form on* P *and* $\Phi_P(x) = \theta(x)$*. In other words, there is a one-to-one correspondence between isomorphism classes of contact manifolds and isomorphism classes of symplectic* $\mathbb{R}^\times$*-bundles.*

We call (P, τ, M, h, ω) a symplectic cover of (M, C).

Examples

Example (Projectivizations of cotangent bundles)

The $\mathbb{R}^\times$-bundle $P = (\mathsf{T}^*M)^\times$ with $\omega = \omega_M\big|_P$ is a symplectic $\mathbb{R}^\times$-bundle. It represents the canonical contact structure on the projectivization of T^*M,

$$P/\mathbb{R}^\times \simeq \mathbb{P}(\mathsf{T}^*M)\,,$$

which is non-trivializable if $\dim(M)$ is odd.

Example (Extended cotangent bundle)

A canonical symplectic cover of the canonical contact structure

$$\eta(z,\mathbf{p},q) = \mathrm{d}z - \theta_Q(\mathbf{p},q)$$

on $M = \mathbb{R}^* \times \mathsf{T}^*Q$ is $P = \mathsf{T}^*(\mathbb{R}^\times \times Q) = \mathbb{R}^\times \times M$ with $\omega = \omega_{\mathbb{R}^\times \times Q}$ and

$$\tau(s,q^i,z,p_j) = (z,\mathbf{p}_j = p_j/s, q^i).$$

Contact – symplectic correspondence

Theorem

Let $(P_i, \tau_i, M_i, h^i, \omega_i)$ *be a symplectic cover of a contact structure* (M_i, C_i), $i = 1, 2$. *Every isomorphism* $\widetilde{\varphi} : P_1 \to P_2$ *of symplectic* $\mathbb{R}^\times$*-bundles covers a unique contactomorphism* $\varphi : M_1 \to M_2$ *of the corresponding contact manifolds.*

Conversely, any contactomorphism $\varphi : M_1 \to M_2$ *is covered by a unique isomorphism*

$$\begin{array}{ccc} P_1 & \xrightarrow{\widetilde{\varphi}} & P_2 \\ {\scriptstyle \tau_1}\downarrow & & \downarrow{\scriptstyle \tau_2} \\ M_1 & \xrightarrow{\varphi} & M_2 \end{array}$$

of symplectic $\mathbb{R}^\times$*-bundles.*

Contact – symplectic dictionary

(P, τ, M, h, ω) - a symplectic cover of a contact mfd (M, C)

contact Hamiltonian $\longleftrightarrow$ 1-homogeneous Hamiltonian on P
or section $\sigma : M \to L_C = \mathsf{T}M/C$, $H = \iota_\sigma$, $\sigma = \sigma_H$;

For trivial $P = \mathbb{R}^\times \times M$: $\boxed{H(s,x) = s\,\hat{H}(x)}$ $\to$ Physics;

contact vector field X^c on M $\longleftrightarrow$ $\boxed{X^c = \tau_*(X_H)}$
$\mathbb{R}^\times$-invariant Hamiltonian vector field X_H on P;

contact-Jacobi bracket on L_C $\longleftrightarrow$ Poisson bracket of
1-homogeneous Hamiltonians, $\boxed{\{\sigma_H, \sigma_{H'}\}_C = \sigma_{\{H,H'\}_\omega}}$;

Contact – symplectic dictionary

submanifold $N \subset M \longleftrightarrow \mathbb{R}^\times$-subbundle $\widetilde{N} = \tau^{-1}(N)$

N - (co)isotropic $\longleftrightarrow \widetilde{N}$ - (co)isotropic;

N - Legendrian $\longleftrightarrow \widetilde{N}$ - Lagrangian;

contact H-J equation on $M = \mathbb{R}^* \times \mathsf{T}^*Q$:

$\hat{H} : M \to \mathbb{R}$, $\sigma : Q \to \mathbb{R}^*$,
$S(s,q) = s\sigma(q)$ - function on $\mathbb{R}^\times \times Q$,

$H : \mathsf{T}^*(\mathbb{R}^\times \times Q) \to \mathbb{R}$, $H(s,q,z,p) = s\hat{H}(z, p/s, q)$,

$\boxed{\text{classical H-J eq.: } H \circ \mathrm{d}S = 0} \Leftrightarrow \boxed{\hat{H} \circ \mathsf{j}^1(\sigma) = 0}$.

Linear contact structures

For a line bundle $\tau_0 : L \to Q$, consider $P = \mathsf{T}^* L^\times$. It is canonically symplectic $\mathbb{R}^\times$-bundle with $\omega = \omega_{L^\times}$ and the lifted $\mathbb{R}^\times$-action

$$(\mathrm{d}_{\mathsf{T}^*} h)_s = s \cdot (\mathsf{T} h_{s^{-1}})^*.$$

It represents the canonical contact structure on the bundle $\mathsf{T}^* L^\times / \mathbb{R}^\times = \mathsf{J}^1 L^*$ of first jets of sections of the dual bundle L^*. This contact structure is trivializable iff L trivializable.

$$\begin{array}{ccc}
\mathsf{T}^*(L^\times) & \xrightarrow{\ \tau\ } & \mathsf{J}^1(L^*) \\
\Big\downarrow {\scriptstyle \pi_{L^\times}} & & \Big\downarrow {\scriptstyle \tau^1(L^*)} \\
L^\times & \xrightarrow{\ \tau_0\ } & Q\,.
\end{array}$$

[G2013] $\boxed{\mathsf{J}^1 L \text{ are the only linear contact manifolds}}$.

Group actions

(P, τ, M, h, ω) - a symplectic cover of (M, C)

There is a one-to-one correspondence between actions

$$\rho : G \times M \to M, \quad \rho(g, y) = g^c(y),$$

of a Lie group G on M by contactomorphisms and actions

$$\widetilde{\rho} : G \times P \to P, \quad \widetilde{\rho}(g, x) = \widetilde{g}(x)$$

of G on P by $\mathbb{R}^\times$-equivariant symplectomorphisms.

This induces a homomorphism $\xi \mapsto \xi^c$ of the Lie algebra $\mathfrak{g}$ of G into the Lie algebra of contact vector fields on M.

ξ^c is covered by $\hat{\xi}$ - an $\mathbb{R}^\times$-invariant Hamiltonian vector field $\hat{\xi}$ on P with 1-homogeneous Hamiltonian $H_\xi = i_{\hat{\xi}}\theta$ on P.

Contact moment maps

The canonical *contact moment map* is

$$J : P \to \mathfrak{g}^* , \quad \langle J(x), \xi \rangle = H_\xi(x) = (i_{\hat{\xi}}\, \theta)(x).$$

J is Ad-equivariant and $\mathbb{R}^\times$-equivariant, $J(h_s(x)) = s \cdot J(x)$.

We apply a modified Marsden-Weinstein-Meyer reduction for a weakly regular value μ of J (G-connected).

$P_{[\mu]} = J^{-1}([\mu]^\times) \simeq \mathbb{R}^\times \times J^{-1}(\mu)$;

$G_\mu = \{g \in G :\ \mathrm{Ad}^*_g(\mu) = \mu\}$ - the stabilizer subgroup;

$\mathfrak{g}^0_\mu = \{v \in \ker(\mu) :\ \mathrm{ad}^*_v(\mu) = 0\}$ is a Lie ideal in $\mathfrak{g}_\mu$;

Contact Marsden-Weinstein reduction

G^0_μ the corresponding Lie subgroup of G - acts on $P_{[\mu]}$;

$M_\mu = \tau(P_{[\mu]}) = \tau(J^{-1}(\mu))$ - submanifold in M.

Theorem (Contact reduction)

Suppose that M_μ is transversal, i.e., $\mathsf{T}M_\mu \cap C$ is of corank 1 in $\mathsf{T}M_\mu$.

If, moreover, G^0_μ is a closed subgroup and acts freely and properly on $P_{[\mu]}$, then $P_{red} = P_{[\mu]}/G^0_\mu$ is a symplectic $\mathbb{R}^\times$-bundle.
It covers the manifold $M_{red} = M_\mu/G^0_\mu$ which is equipped with the contact structure $C_{red} = \mathsf{T}p(\mathsf{T}M_\mu \cap C)$, where $p : M_\mu \to M_\mu/G^0_\mu$ is the canonical surjective submersion.

When G^0_μ is closed?

Theorem

If G *is compact and connected, then* G^0_μ *is closed in* G_μ*, thus in* G*, if and only if* μ *is integral, i.e., there exists* $\hbar \in \mathbb{R}^\times$ *such that*

$$\varphi : G_\mu \to S^1, \quad \varphi(\exp(\xi)) = e^{2\pi\hbar\, i \langle \mu, \xi \rangle},$$

is a well-defined unitary character on G_μ*.*

This, in turn, is equivalent to the fact that the canonical symplectic form (KKS) $\omega_{\mathcal{O}_\mu}$ *on the coadjoint orbit* $\mathcal{O}_\mu$ *is integral,* $\left[(2\pi\hbar)^{-1}\omega_{\mathcal{O}_\mu}\right] \in H^2(\mathcal{O}_\mu; \mathbb{Z})$*.*

contact geometry $\leftrightarrow$ geometric prequantization

Example

A standard symplectic reduction: G acts freely and properly on M. This action can be lifted to a Hamiltonian G-action on T^*M with a moment map $J : \mathsf{T}^*M \to \mathfrak{g}^*$. Then,

$$J^{-1}(0)/G \simeq \mathsf{T}^*(M/G).$$

A contact analogue: Let $\tau_0 : L \to Q$ be a line bundle. Any G-action on L by vector bundle automorphisms can be canonically lifted to a contact G-action on J^1L^*. The Hamiltonian action of G on the symplectic cover $\mathsf{T}^*L^\times$ of J^1L^* is the standard cotangent lift of the G-action on $L^\times$ and $J : \mathsf{T}^*L^\times \to \mathfrak{g}^*$. Assuming the G-action on L is free and proper, we get the reduced symplectic $\mathbb{R}^\times$-bundle

$$J^{-1}(0)/G = \mathsf{T}^*(L^\times/G) = \mathsf{T}^*(L/G)^\times,$$

and the reduced contact manifold is

$$\mathsf{J}^1(L/G)^* = \mathsf{J}^1(L^*/G).$$

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THANK YOU FOR YOUR ATTENTION!