Banach Lie groupoids, algebroids, von Neumann algebras and restricted Grassmannian

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Anatol Odzijewicz and Aneta Sliżewska.
Banach–Lie groupoids associated to W^*-algebras.
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Anatol Odzijewicz, Grzegorz Jakimowicz, and Aneta Sliżewska.
Banach–Lie algebroids associated to the groupoid of partially invertible elements of a W^*-algebra.
J. Geom. Phys., 95:108–126, 2015.

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Fiber-wise linear Poisson structures related to W-algebras.
J. Geom. Phys., 123:385–423, 2018.

Daniel Beltiţă, Tomasz Goliński, Grzegorz Jakimowicz, and Fernand Pelletier.
Banach–Lie groupoids and generalized inversion.
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Tomasz Goliński, Grzegorz Jakimowicz, and Aneta Sliżewska.
Banach–Lie groupoid of partial isometries over restricted Grassmannian.
to appear.

Some problems in infinite dimensional geometry

- No dimension counting arguments: existence of injective automorphisms which are not surjective.
- Closed subspaces may not be complemented.
- Many classical theorems fail or require non-trivial modifications, e.g. there is no Hopf–Rinow theorem, Frobenius requires extra assumptions.
- Lack of partitions of unity or bump functions: no localization
- Double dual of the a Banach space may not be canonically isomorphic to the original space.

Let us recall the definition of Banach Lie groupoid in the Hausdorff case.

• Submersion: a smooth map $f : N \to M$ between two Banach manifolds such that for each $x \in N$ the tangent map $Tf : T_xN \to T_{f(x)}M$ is a surjection and $\ker T_xf$ is a split subspace of T_xN.

Definition

A Banach–Lie groupoid $\mathcal{G} \rightrightarrows M$: a pair $(\mathcal{G}, M)$ of Banach manifolds and maps:

- surjective submersions $s : \mathcal{G} \to M$ and $t : \mathcal{G} \to M$ (*source* and *target*)
- smooth map $m : \mathcal{G}^{(2)} \to \mathcal{G}$, where $\mathcal{G}^{(2)} := \{(g,h) \in \mathcal{G} \times \mathcal{G} \mid s(g) = t(h)\}$ with induced topology (*multiplication* $m(g,h) = gh$) and satisfying associativity condition in the sense that the product $(gh)k$ is defined if and only if $g(hk)$ is defined and in this case they coincide.
- continuous embedding $\varepsilon : M \to \mathcal{G}$ (*identity section*) such that $g\varepsilon(x) = g$ for all $g \in s^{-1}(x)$, and $\varepsilon(x)g = g$ for all $g \in t^{-1}(x)$.
- diffeomorphism $\iota : \mathcal{G} \to \mathcal{G}$, (*inversion*), which satisfies $g\iota(g) = \varepsilon(t(g))$, $\iota(g)g = \varepsilon(s(g))$ for all $g \in \mathcal{G}$.

- $\mathcal{G}^{(2)}$ is a submanifold of $\mathcal{G} \times \mathcal{G}$
- map ε is smooth
- $\varepsilon(M)$ is a closed Banach submanifold of $\mathcal{G}$

For a W^*-algebra $\mathfrak{M}$ one constructs two groupoids:

- partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$
- partially invertible operators $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$
- base in both cases is the manifold of projectors $\mathcal{L}(\mathfrak{M})$.
- algebroids and related Poisson structures were also described.

- Manifold of projectors

$$\mathcal{L}(\mathfrak{M}) = \{p \in \mathfrak{M} \mid p^2 = p^* = p\}$$

- Groupoid of partial isometries $\mathcal{U}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

$$\mathcal{U}(\mathfrak{M}) = \{u \in \mathfrak{M} \mid uu^*u = u\}$$

- Groupoid of partially invertible operators $\mathcal{G}(\mathfrak{M}) \rightrightarrows \mathcal{L}(\mathfrak{M})$

$$\mathcal{G}(\mathfrak{M}) = \{x \in \mathfrak{M} \mid \ |x| \in G(\operatorname{supp}|x|\,\mathfrak{M}\operatorname{supp}|x|)\}$$

- $\operatorname{supp}(x)$ — support of an element x

The groupoid structure on $\mathcal{U}(\mathfrak{M})$ is given by:

$$\begin{aligned} s(u) &= u^* u \\ t(u) &= u u^* \\ m(u_1, u_2) &= u_1 u_2 \\ \varepsilon(p) &= p \\ \iota(u) &= u^* \end{aligned}$$

for $u, u_1, u_2 \in \mathcal{U}$ such that $u_2 u_2^* = u_1^* u_1$ and $p \in \mathcal{L}(\mathcal{H})$.

For $\mathcal{G}(\mathfrak{M})$ s and t are right and left supports and $\iota(x)$ is expressed by an inverse element in $G(\sigma(|x|)\mathfrak{M}\sigma(|x|))$ using polar decomposition.

$$
\begin{array}{ccc}
\mathcal{U}(\mathfrak{M}) & \hookrightarrow & \mathcal{G}(\mathfrak{M}) \\
s \downarrow\downarrow t & & s \downarrow\downarrow t \\
\mathcal{L}(\mathfrak{M}) & \xrightarrow{id} & \mathcal{L}(\mathfrak{M})
\end{array}
$$

MANIFOLD STRUCTURE ON $\mathcal{L}(\mathfrak{M})$

- For $p \in \mathcal{L}(\mathfrak{M})$ we define

$$\Pi_p := \{q \in \mathcal{L}(\mathfrak{M}) \ : \ \mathfrak{M} = q\mathfrak{M} \oplus (1-p)\mathfrak{M}\}$$

- Decomposition $p = x_p - y_p \ \in q\mathfrak{M} \oplus (1-p)\mathfrak{M}$ defines the maps

$$\sigma_p : \Pi_p \to \ q\mathfrak{M}p, \qquad \varphi_p : \Pi_p \ \to \ (1-p)\mathfrak{M}p$$

$$\sigma_p(q) := x_p, \qquad \varphi_p(q) := y_p$$

- φ_p is a bijection

Proposition

The charts

$$(\Pi_p, \varphi_p) \quad p \in \mathcal{L}(\mathfrak{M})$$

defines a complex analytic atlas on $\mathcal{L}(\mathfrak{M})$. This atlas is modeled on the Banach spaces $(1-p)\mathfrak{M}p$.

The transitions maps

$$\varphi_p \circ \varphi_{p'}^{-1} : \varphi_{p'}(\Pi_p \cap \Pi_{p'}) \to \varphi_p(\Pi_p \cap \Pi_{p'})$$

between $y_p = \varphi_p(q)$ and $y_{p'} = \varphi_{p'}(q)$ is given by

$$y_{p'} = (\varphi_{p'} \circ \varphi_p^{-1})(y_p) = (b + dy_p)(a + cy_p)^{-1},$$

where $q \in \Pi_p \cap \Pi_{p'}$ and $a = p'p$, $b = (1-p')p$, $c = p'(1-p)$, $d = (1-p')(1-p)$.

Manifold structure on $\mathcal{G}(\mathfrak{M})$

- For projections $\tilde{p}, p \in \mathcal{L}(\mathfrak{M})$ we define

$$\Omega_{\tilde{p}p} := t^{-1}(\Pi_{\tilde{p}}) \cap s^{-1}(\Pi_p) \subset \mathcal{G}(\mathfrak{M})$$

- The maps

$$\psi_{\tilde{p}p} : \Omega_{\tilde{p}p} \to (1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

$$\psi_{\tilde{p}p}(x) := \big(\varphi_{\tilde{p}}(t(x)), (\sigma_{\tilde{p}}(t(x)))^{-1} x \sigma_p(s(x)), \varphi_p(s(x))\big) = (y_{\tilde{p}}, z_{\tilde{p}p}, y_p)\,.$$

Proposition

The charts

$$(\Omega_{\tilde{p}p}, \psi_{\tilde{p}p}) \quad \tilde{p}, p \in \mathcal{L}(\mathfrak{M})$$

define a complex analytic atlas on the groupoid $\mathcal{G}(\mathfrak{M})$. *with modeling Banach spaces*

$$(1-\tilde{p})\mathfrak{M}\tilde{p} \oplus \tilde{p}\mathfrak{M}p \oplus (1-p)\mathfrak{M}p$$

indexed by the pair of projections from $\mathcal{L}(\mathfrak{M})$.

Proposition

The groupoid $\mathcal{G}(\mathfrak{M}) \longrightarrow \mathcal{L}(\mathfrak{M})$ *equipped with the above Banach manifold structure is a complex Banach Lie groupoid.*

Remark

Banach Lie groupoid structure on $\mathcal{U}(\mathfrak{M}) \longrightarrow \mathcal{L}(\mathfrak{M})$ is real part of the described structure.

Remark

Banach Lie algebroid counterparts to these groupoids were also described.

Remark

The set of partial isometries has a Banach Lie groupoid structure also in the case of a C^*-algebra.

Remark

Partially invertible elements in C^*-algebra are also known as Moore-Penrose invertible elements. To our best knowledge there is no Banach Lie groupoid structure there due to discontinuity of inversion map.

Particular case: $\mathfrak{M} = L^\infty(\mathcal{H})$

• differential structure on $\mathcal{L}(\mathcal{H})$: inverse map $\varphi_W^{-1} : L^\infty(W, W^\perp) \to \mathcal{L}(\mathcal{H})$ to a chart assigns to a bounded operator its graph in $W \oplus W^\perp = \mathcal{H}$

$$\varphi_W^{-1}(A) = \{(w, Aw) \in W \times W^\perp \mid w \in W\}$$

• charts $\varphi_W : U_W \to L^\infty(W, W^\perp)$

$$\varphi_W(V) = P_{W^\perp}(P_W|_V)^{-1}$$

$$U_W = \{V \in \mathcal{L}(\mathcal{H}) \mid V \oplus_B W^\perp = \mathcal{H}\}$$

Restricted Grassmannian and groupoids

$$\begin{array}{ccccc} \mathcal{U}_{\mathrm{res}} & \hookrightarrow & \mathcal{U}(\mathcal{H}) & \hookrightarrow & \mathcal{G}(\mathcal{H}) \\ s \downdownarrows t & & s \downdownarrows t & & s \downdownarrows t \\ \mathrm{Gr}_{\mathrm{res}} & \hookrightarrow & \mathcal{L}(\mathcal{H}) & \xrightarrow{id} & \mathcal{L}(\mathcal{H}) \end{array}$$

• fix an orthogonal decomposition (called polarization) of the Hilbert space $\mathcal{H}$

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

onto infinite dimensional Hilbert subspaces $\mathcal{H}_\pm$.

• P_+, P_-: the orthogonal projectors onto $\mathcal{H}_+$ and $\mathcal{H}_-$

• block decomposition of an operator A acting on $\mathcal{H}$:

$$A = \begin{pmatrix} A_{++} & A_{+-} \\ A_{-+} & A_{--} \end{pmatrix}$$

Definition

The **restricted Grassmannian** $\mathrm{Gr}_{\mathrm{res}}$ is defined as a set of Hilbert subspaces $W \subset \mathcal{H}$ such that:

- the orthogonal projection $p_+ : W \to \mathcal{H}_+$ is a Fredholm operator;
- the orthogonal projection $p_- : W \to \mathcal{H}_-$ is a Hilbert–Schmidt operator.

• identify the Hilbert subspace W with a projector P_W onto this subspace.

Proposition

$$W \in \mathrm{Gr}_{\mathrm{res}} \Longleftrightarrow P_W - P_+ \in L^2$$

• Banach Lie group: unitary restricted group U_{res} acting transitively on $\mathrm{Gr}_{\mathrm{res}}$:

$$U_{\mathrm{res}} := \{u \in U(H) \mid [u, P_+] \in L^2\}$$

• its Banach Lie algebra

$$\mathfrak{u}_{\mathrm{res}} := \{\mu \in \mathfrak{u}(\mathcal{H}) \mid [\mu, P_+] \in L^2\} \tag{1}$$

• $\mathrm{Gr}_{\mathrm{res}}$ can be seen as a smooth homogenous space $U_{\mathrm{res}}/(U_+ \times U_-)$

• differential structure on $\mathrm{Gr}_{\mathrm{res}}$ is obtained using the same charts as for $\mathcal{L}(\mathcal{H})$ taking value in $L^2(W, W^\perp)$ and transition functions are smooth in L^2 topology

• for particular case of $W = \mathcal{H}_+$ the inverse to a chart is

$$\varphi_{\mathcal{H}_+}^{-1}(A) = \begin{pmatrix} (1 + A^*A)^{-1} & (1 + A^*A)^{-1}A^* \\ A(1 + A^*A)^{-1} & A(1 + A^*A)^{-1}A^* \end{pmatrix}$$

• equivalently:

$$\varphi_{\mathcal{H}_+}^{-1}(A) = (P_+ + A)(1 + A^*A)(P_+ + A^*P_-)$$

$$\varphi_W^{-1}(A) = (P_W + A)(1 + A^*A)(P_W + A^*P_{W^\perp})$$

One defines a groupoid $\mathcal{U}_{\text{res}} \rightrightarrows \text{Gr}_{\text{res}}$ as

$$\mathcal{U}_{\text{res}} = s^{-1}(\text{Gr}_{\text{res}}) \cap t^{-1}(\text{Gr}_{\text{res}})$$

$$= \{u \in \mathcal{U}(\mathcal{H}) \mid u^*u, uu^* \in \text{Gr}_{\text{res}}\}$$

- It is a subgroupoid (in algebraic sense)

Proposition

For $u \in \mathcal{U}_{res}$ we have $u_{+-}, u_{-+} \in L^2$.

Proposition

For every point $W \in \mathrm{Gr}_{\mathrm{res}}$ there exists a neighbourhood $\Omega_W \subset \mathrm{Gr}_{\mathrm{res}}$ and a smooth map $\sigma_W : \Omega_W \to U_{res}$ such that

$$\forall\ W' \in \Omega_W \quad W' = \sigma_W(W')\mathcal{H}_+$$

- Using these local cross sections we construct injective maps:

$$\mathcal{U}_{\mathrm{res}} \supset s^{-1}(W') \cap t^{-1}(W) \longrightarrow \mathrm{Gr}_{\mathrm{res}} \times U_+ \times \mathrm{Gr}_{\mathrm{res}}$$

$$u \mapsto (uu^*, \sigma_W(uu^*)^{-1} u \sigma_{W'}(u^* u)_{|\mathcal{H}_+}, u^* u)$$

- couple this map with charts on the respective manifolds

$$\Phi_{\alpha\beta\gamma}(u) = (\tilde{\psi}_\gamma(uu^*), \tilde{\psi}_\beta(u^* u), \psi_\alpha(\sigma_\gamma(uu^*)^{-1} u \sigma_\beta(u^* u))_{|\mathcal{H}_+})$$
$$\in L^2(W_\gamma, W_\gamma^\perp) \times L^2(W_\beta, W_\beta^\perp) \times \mathfrak{u}(\mathcal{H})_+$$

Theorem

The family $(\Omega_{\alpha\beta\gamma}, \Phi_{\alpha\beta\gamma})$ *constitutes a smooth atlas on* $\mathcal{U}_{res}$.

Theorem

The manifold $\mathcal{U}_{res}$ *is a Banach–Lie groupoid with respect to the defined maps.*

• Obviously, $\mathcal{U}_{\rm res}$ is not a Banach–Lie subgroupoid of the Banach–Lie groupoid of all partial isometries $\mathcal{U}(\mathcal{H})$: Grassmannian $\mathcal{L}(\mathcal{H})$ is not a submanifold of the restricted Grassmannian $\mathrm{Gr}_{\rm res}$. It is only a weakly immersed submanifolds as the image of the tangent of the inclusion map is not closed.

• Unlike $\mathcal{U}(\mathcal{H})$, the groupoid $\mathcal{U}_{\rm res}$ is transitive and pure, i.e. the map $(s,t) : \mathcal{U}_{\rm res} \to \mathrm{Gr}_{\rm res} \times \mathrm{Gr}_{\rm res}$ is surjective and both base and total space are modeled on a single (up to isomorphism) Banach space.