A quadric of kinetic energy in the role of phase diagrams
Application to the BKL scenario

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Outline

1. Role of the phase diagrams in first-order ODE and difficulties with generalization to more dimensions
2. The example: Belinski-Khalatnikov-Lifshitz (BKL) scenario
   - From Einstein equations in a neighbourhood of the cosmological singularity to the BKL equations
   - Symmetries of the BKL equations
   - Exact solution of the BKL system and its instability
   - Hamiltonian approach to the BKL equations
3. Cone of the kinetic energy as a tool for analysing dynamics
   - Reading the dynamics from the cone
   - Image of the exact solution and its small perturbations
   - Other solutions' asymptotics for large time; their stability
   - Numerical simulations
4. Conclusions

Phase diagrams for first-order ordinary differential equations (ODE)

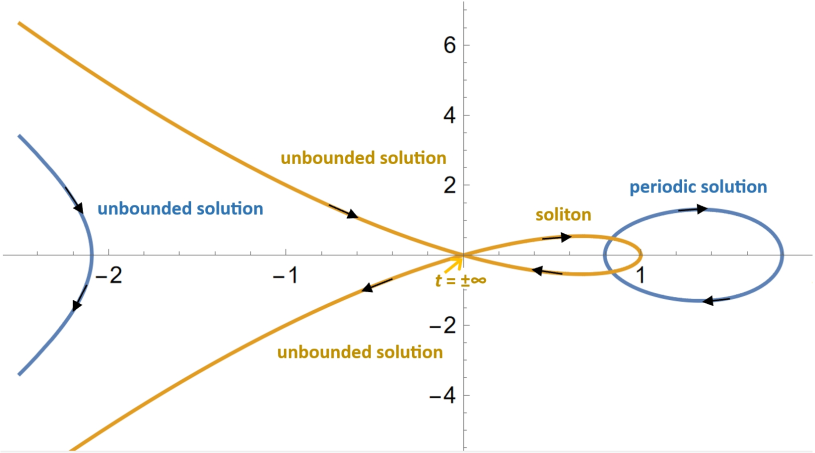

Figure: Possible phase diagrams for a twice integrated travelling-wave reduction of the KdV equation $-Vu^2/2 + u^3 + {u_x}^2/2 + c_1 u + c_2 = 0$, where V is the propagation velocity of the wave, while c_1 and c_2 are integration constants. The self-intersecting loop ($c_1 = c_2 = 0$) represents a soliton and 2 unbounded solutions (although the orange line is connected, it consists of 3 different solutions).

Information we get from phase diagrams

1. Type of the solution (solitary, periodic, bounded, unbounded, etc.).
2. Minima and maxima of the dependent variable u and also of its derivative u_t.
3. Direction of growth of the independent variable, from $dt = du/u_t$.
4. A check whether the range of the independent variable in a given loop is finite or infinite.
5. When the range is infinite, we can identify the limits of the solution at infinity.
6. We can see if there are several distinct solutions for the same value of the parameters.

Some of this information may be obtained for Hamiltonian systems from the quadric representing their kinetic part.

Example: the Belinski-Khalatnikov-Lifshitz scenario

- The scenario is an answer to the question whether a generic singularity exists in cosmology.
- Generic not in the sense that it has to exist in most situations, rather:
- The singularity occurs for a subset of nonzero measure in the space of initial conditions of the proper dimensionality (with the proper number of arbitrary constants).

Vladimir Alekseevich Belinski (the only living one)

Isaak Markovich Khalatnikov

Evgeny Mikhailovich Lifshitz

Belinski-Khalatnikov-Lifshitz (BKL) scenario

- Describes the universe in a neighbourhood of the cosmic singularity *[V. A. Belinskii, I. M. Khalatnikov, and E. M. Lifshitz, 1970, Adv. Phys.,* **19** *(80), 525.]*
- Contrary to the previous models, it allows for anisotropy.
- The anisotropy is measured by the ratios of length scales a, b, c ("scale factors") in three principal directions.
- With direction-dependent rate of collapse, it is likely, and thus assumed, that the anisotropy increases indefinitely as we approach the singularity.
- Question: how do the scale factors and principal directions vary when we approach the singularity?

From 10 Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} R\, g_{\alpha\beta} = \kappa T_{\alpha\beta}$$

in a neighbourhood of the cosmological singularity to 3 BKL equations

($R_{\alpha\beta}$ = Ricci tensor, $\alpha, \beta = 1, 2, 3, 4$, R = Ricci scalar, $g_{\alpha\beta}$ = 4-dimensional metric tensor, $\kappa = G/(8\pi c^4)$, G = gravitational constant $T_{\alpha\beta}$ = stress-energy tensor)

Shortly, on the derivation of the BKL equations

1. Neglecting the influence of matter on the metric, $T_{\mu\nu} = 0$.
2. Synchronous frame of reference: at each point, time is the proper time (co-moving observer) $\rightarrow$ 3-dim Bianchi models.
3. Rescaling the time to $t/\sqrt{\gamma}$; γ is the spatial metric tensor.
4. Using the Bianchi identities to once integrate the equations for the off-diagonal Ricci tensor.
5. Specifying the spatial directions of the frame axes.
6. Diagonalising the metric to $\mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3)$, $\Gamma_1 > \Gamma_2 > \Gamma_3$, with principal axes rotating in this frame.
7. Going to the limit $\Gamma_2/\Gamma_1 \rightarrow 0$, $\Gamma_3/\Gamma_2 \rightarrow 0$.
8. Rescaling $\Gamma_1, \Gamma_2, \Gamma_3$, by constants of order 1, to *a*, *b*, *c*.

The result consists of 3 equations for the evolution of *a*, *b*, *c* with 1 constraint equation, while 3 equations for the Euler angles in this frame show that the rotation stops.

Three spatial-temporal Einstein equations do not influence the dynamics.

A sketch of the derivation of the BKL equations

(7 slides omitted at the conference)

- First choose the frame of reference which provides the simplest description.

- The synchronous frame: the time is the proper time of an observer co-moving with the observed object. Then the metric form

$$ds^2 = dt^2 - \gamma_{ab}(t)dx^a dx^b$$

(summation convention, the spatial metric tensor $\gamma_{ab} = -g_{ab}$, where a, b=1,2,3, depends on time only).

- This reduces the Einstein equations to a problem of finding the appropriate 3D metric; the corresponding Lie algebra structure has been classified by Bianchi. The most general, anisotropic, is Bianchi IX; the structure constants may be ϵ_{abc} (ϵ is the Levi-Civita symbol).

Simplifications

- Rescaling dt^2 by the spatial volume

$$dt = \sqrt{\gamma}dt',$$

where γ is the determinant of the spatial (time-dependent) metric tensor γ_{ab}. In what follows we will omit the prime.

- Close to the singularity, the stress-energy tensor is negligible compared to the (singular) Ricci tensor components.

- Of ten Einstein equations, the ${}^{0}_{a}$ components ($a = 1, 2, 3$) provide only relations between constants, they do not describe the dynamics. What remains are six equations for $R^b_a, \quad (a, b = 1, 2, 3),$ and one for R^0_0.

The intermediate system of equations

- The six Einstein equations for the spatial components (dot = differentiation with respect to the rescaled time) read

$$R_a^b = \frac{1}{2\gamma}\dot{k}_a^b + P_a^b = 0,$$

- and the one for the temporal component has the form

$$R_0^0 - R_a^a = \frac{1}{4\gamma}\left(k_a^b k_b^a - \frac{(\dot{\gamma})^2}{\gamma^2}\right) - P_a^a = 0,$$

where $k_a^b = \gamma^{bc}\dot{\gamma}_{ca}$ and P_a^b are components of the 3D Ricci tensor (summation over identical indices applies).

Further specification of the frame

- A possibility of integration of the off-diagonal equations follows from the Bianchi identities, yielding

$$\epsilon_{abc} k_a^b = C_c = const.,$$

where C_c is a vector integral of motion.

- The choice of the synchronous frame is not unique. E.g. we have freedom of its rotation.

- We use constants of motion C_c to fix its orientation

- Our choice: $C_1 = C_2 = 0, \quad C_3 =: C$

The system in the special frame

- The spatial metric tensor $\hat{\gamma}$ may be diagonalised and it has its principal axes in the 3D space.
- Let the diagonalised matrix be

$$\hat{\Gamma} = \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3) \, .$$

- In general, the axes would rotate with respect to our chosen frame.
- The classical description for a rotating system is in terms of the Euler angles.
- The total rotation $\hat{\mathcal{R}}$ is a composition (group product):

$$\hat{\mathcal{R}} = \hat{\mathcal{R}}_\psi \hat{\mathcal{R}}_\theta \hat{\mathcal{R}}_\varphi$$

of 3 rotations: by angle φ about the *z* axis (precession angle), by θ about the *x* axis (inclination or nutation angle) and by ψ about the new *z* axis (pure rotation angle).

Dynamics of rotation

- The spatial equations for the off-diagonal components of the Ricci tensor determine the rotation

$$\sin\,\theta\,\sin\psi\,\dot{\varphi}+\cos\,\psi\,\dot{\theta}=\frac{\Gamma_2\Gamma_3\sin\,\theta\,\sin\,\psi}{(\Gamma_2-\Gamma_3)^2}$$

$$\sin\,\theta\,\cos\,\psi\,\dot{\varphi}-\sin\,\psi\,\dot{\theta}=\frac{\Gamma_3\Gamma_1\sin\,\theta\,\cos\,\psi}{(\Gamma_3-\Gamma_1)^2}$$

$$\cos\,\theta\,\dot{\varphi}+\dot{\psi}=\frac{\Gamma_1\Gamma_2\sin\,\theta\,\cos\,\psi}{(\Gamma_1-\Gamma_2)^2}.$$

- BKL conjecture: the initial anisotropy of the system should indefinitely grow.
- We number Γ's by their order: $\Gamma_1 > \Gamma_2 > \Gamma_3$. Then, close to the singularity, we have $\Gamma_1 \gg \Gamma_2 \gg \Gamma_3$ and may neglect terms proportional to Γ_3/Γ_2, Γ_2/Γ_1, and Γ_3/Γ_1,
- then the t-derivatives of the angles vanish in the zero order $\Longrightarrow$ rotation of the principal axes stops close to the singularity: $(\theta,\varphi,\psi)\to(\theta_0,\varphi_0,\psi_0)$.

Dynamics of the scale factors

- The other 3 equations describe the dynamics of three Γ's. They correspond to the diagonal spatial Einstein equations (*omitted here because of their complexity*).

- The last one is the Einstein equation for the temporal (diagonal) component R_0^0 (*omitted for the same reason*).

- With these equations, go to the limit $\Gamma_3/\Gamma_2 \to 0, \ \Gamma_2/\Gamma_1 \to 0$,

- Rescale the Γ's to define the new variables a, b, and c

$$\Gamma_1 =: a, \quad \Gamma_2 C^2 \cos^2\theta_0 =: b, \quad \Gamma_3 C^4 \sin^2\theta_0 \cos^2\theta_0 \sin^2\varphi_0 =: c.$$

The Belinski-Khalatnikov-Lifshitz equations

The equations

- The three equations corresponding to the diagonal spatial components become

$$\boxed{\frac{d^2 \ln a}{dt^2} = \frac{b}{a} - a^2} \quad \boxed{\frac{d^2 \ln b}{dt^2} = a^2 - \frac{b}{a} + \frac{c}{b}} \quad \boxed{\frac{d^2 \ln c}{dt^2} = a^2 - \frac{c}{b}},$$

- while the temporal component yields a constraint

$$\boxed{\frac{d \ln a}{dt}\frac{d \ln b}{dt} + \frac{d \ln a}{dt}\frac{d \ln c}{dt} + \frac{d \ln b}{dt}\frac{d \ln c}{dt} = a^2 + \frac{b}{a} + \frac{c}{b}}.$$

This is the BKL system.

- Apparently overdetermined (4 equations, 3 unknowns), it is in fact consistent. Namely, the 3rd equation may be obtained by substitution of $\ddot{a}$ and $\ddot{b}$ from the first two to the t-derivative of the constraint: it yields the 3rd equation multiplied by $(\dot{a}/a + \dot{b}/b)$ *(dot denotes time differentiation)*.

- It describes a collapse of the homogeneous but anisotropic universe or (by time-reversal) its emerging in an explosion, for the time close to the cosmological singularity.
- The authors (BKL) conjecture that the approach to the singularity is chaotic.
- We will show that it may describe oscillations to the singularity and back, which stems from the natural assumptions imposed on the Einstein equations, (rather than a presumed form of the metric tensor, like e.g. the well-known Kasner's universe, in which $ds^2 = -dt^2 + \sum_{j=1}^{3} t^{2p_j} (dx^j)^2$, where $\sum_{j=1}^{3} p_j = 1$, $\sum_{j=1}^{3} p_j^2 = 1$).
- If $a \to 0$ when $t \to \infty$ (close to the singularity), with t^{-1} being a small quantity of order ε , then in that region $\boxed{a \sim \varepsilon^{1/2}, \quad b \sim \varepsilon^{3/2}, \quad c \sim \varepsilon^{5/2}}$, which is consistent with the assumption $a \gg b \gg c$ (growing anisotropy).

The canonical structure of the BKL system

- BKL is a Lagrangian-Hamiltonian system in the variables $x_1 = \ln a, \quad x_2 = \ln b, \quad x_3 = \ln c$.
  [E. Czuchry and W. Piechocki Phys. Rev. D **87**, *084021 (2013)]*.

- The Hamiltonian is the l.h.s. of the constraint, which corresponds to zero value of the Hamiltonian. The Lagrangian reads
  $$\mathcal{L} = \underbrace{\dot{x}_1\dot{x}_2 + \dot{x}_2\dot{x}_3 + \dot{x}_3\dot{x}_1}_{\text{“kinetic energy” } E_k} - \underbrace{\left(-e^{2x_1} - e^{x_2-x_1} - e^{x_3-x_2}\right)}_{\text{“potential energy” } E_p} .$$
  The Hamiltonian (not yet canonical), $\mathcal{H} = E_k + E_p = 0$

- Since the constraint is imposed on an integral of motion, it may replace one of the dynamic BKL equations.

Symmetries of the BKL system

- It is not symmetric under permutation of the spatial directions, because we have defined a as the largest and c the smallest scale factor, and assumed $a \gg b \gg c$ at its derivation.
- The system is symmetric under the change of the arrow of time $t \rightleftarrows -t$.
- There are only two Lie symmetries:
  - The time shift $t' = t - t_0$. This symmetry is also obvious, as the system is autonomous.
  - The scaling symmetry. If λ is the scaling parameter, $t' = \lambda t, \;\; a' = a/\lambda, \;\; b' = b/\lambda^3, \;\; c' = c/\lambda^5$ is a symmetry.
- There exists a self-similar solution with respect to the scaling; it proves to be identical with our exact solution (on the next slides).

The cone of kinetic energy

The kinetic energy of
$E_k = x_1 x_2 + x_2 x_3 + x_3 x_1$
may be diagonalised by

$$x_1 = u_1 - u_2 - u_3,$$
$$x_2 = u_1 + 2u_3,$$
$$x_3 = u_1 + u_2 - u_3$$

leading to

$$E_k = 3\dot{u}_1^2 - \dot{u}_2^2 - 3\dot{u}_3^2,$$

$E_k > 0$ due to $E_p < 0$ while
$\mathcal{H} = E_p + E_k = 0$

[P. Goldstein and W. Piechocki
Eur. Phys. J. C (2022) ***82****:216]*

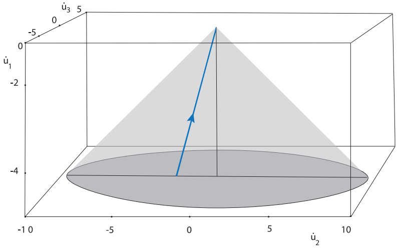

Figure: The lower half (=shrinking volume) of the cone $3\dot{u}_1^2 - \dot{u}_2^2 - 3\dot{u}_3^2 > 0$. The dynamics of the system takes place inside the cone. The blue line shows an exact solution (discussed further; the arrow indicates its direction of evolution). For $t \to \infty$, the line tends to the apex of the cone. The other end of the line extends to infinity (which is beyond the scope of the BKL scenario).

The exact solution

Exact solution of the BKL equations

- We found an exact solution of the BKL equations (no exact solution had been known before)

  [P. Goldstein and W. Piechocki (2022) Eur. Phys. J. C **82***:216]*.

- It reads $\boxed{a(t) = \frac{3}{t-t_0}, \quad b(t) = \frac{30}{(t-t_0)^3}, \quad c(t) = \frac{120}{(t-t_0)^5}.}$

- It is the only meromorphic solution of the system, while the BKL system is non-Painlevé.

- The exact solution consists of the zero-order terms in the Laurent expansion of a, b and c respectively, while all higher order terms prove to be zero from the recurrence relations.

- It is unstable to small perturbations: Although the perturbations tend do zero as $t \to \infty$, their ratios to the respective perturbed quantities grow as $t^{1/2}$.

The exact solution in terms of the diagonal variables

For the variables

$$u_1 = \tfrac{1}{3}\ln[(abc)],$$

$$u_2 = \tfrac{1}{2}\ln[(c/a)],$$

$$u_3 = \tfrac{1}{6}\ln[(b^2/ac)],$$

we have

$$u_1 = \frac{1}{3}\ln\frac{10800}{(t-t_0)^9},$$

$$u_2 = \frac{1}{2}\ln\frac{40}{(t-t_0)^4},$$

$$u_3 = \frac{1}{6}\ln\frac{5}{2} = const.$$

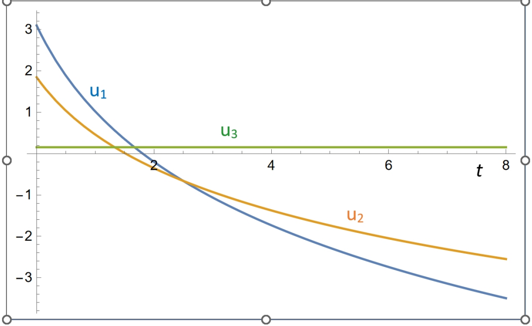

Figure: In the diagonal Lagrangian variables, the exact u_3 is constant, while u_1 and u_2 diverge logarithmically to $-\infty$. In these variables u_1 represents the cubic root of the volume while constancy of u_3 means that b is proportional to $\sqrt{ac}$.

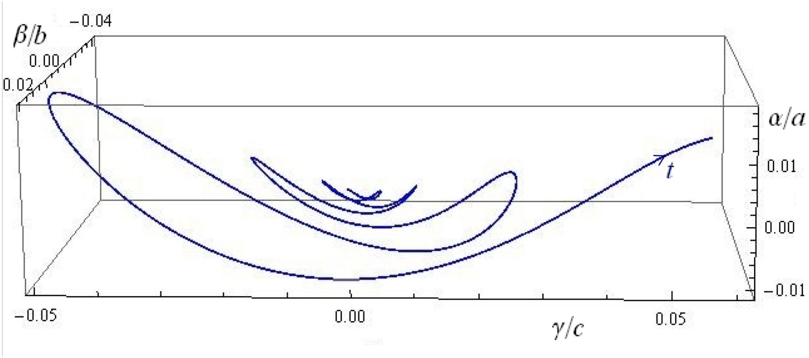

Figure: Linear instability of the exact solution for $K_1 = K_2 = 0.01,\ K_3 = 0, \varphi_1 = \varphi_2 = 0$. The graph presents the parametric curve defined by the time dependence of α/a, β/b, and γ/c, where α, β, γ are the perturbations of a, b, c respectively.

Stochasticity in the neighbourhood of the exact solution

- The phase planes of $(\alpha, \dot{\alpha}), \quad (\beta, \dot{\beta}), \quad (\gamma, \dot{\gamma})$ are ergodically covered by the Lissajous curves of the oscillations with the two incommensurable frequencies.

- On the other hand, the 6D space cannot be densely covered by two-frequency oscillations.

- However the ergodicity of the projections means stochastic behaviour over extended periods of time.

Asymptotic properties of solutions to the BKL equations

The cone of kinetic energy

- The Lagrangian with diagonalised kinetic energy reads

$$\mathcal{L} = \underbrace{3\dot{u}_1^2 - \dot{u}_2^2 - 3\dot{u}_3^2}_{\text{"kinetic energy" } E_k} - \underbrace{\left[-e^{2(u_1-u_2-u_3)} - e^{u_2-3u_3} - e^{u_2+3u_3}\right]}_{\text{"potential energy" } E_p},$$

where $u_1 = \frac{1}{3}\ln(abc)$, $u_2 = \frac{1}{2}\ln(c/a)$, $u_3 = \frac{1}{6}\ln\left(b^2/(ac)\right)$, whence u_1 is the natural logarithm of (volume scale)$^{1/3}$.

- This way, dynamics of the volume is naturally separated from the deformation degrees of freedom.
- The "kinetic part" of the Lagrangian (in orange) is an indefinite quadratic form, whose zero surface in the space of "velocities" $(\dot{u}_1, \dot{u}_2, \dot{u}_3)$ is conical.
- Since the "potential part" (in violet) is always negative, while the "total energy" (Hamiltonian) is zero, then $E_k > 0$ and the evolution takes place inside the cone.

- A position in the cone gives $\dot{u}_1$, $\dot{u}_2$, and $\dot{u}_3$. The tangent to the trajectory consists of $\ddot{u}_1$, $\ddot{u}_2$, $\ddot{u}_3$. Inverting the Lagrange equations, we may get the positions u_1, u_2, u_3.
- Since $u_1 \propto \ln(\text{volume})$, the shrinking universe is represented by the lower half-cone $\dot{u}_1 < 0$.
- The 1st Lagrange equation yields inequality $\ddot{u}_1 = \frac{2}{9} e^{2(u_1 - u_2 - u_3)} > 0$, whence the time arrow is directed up the cone.

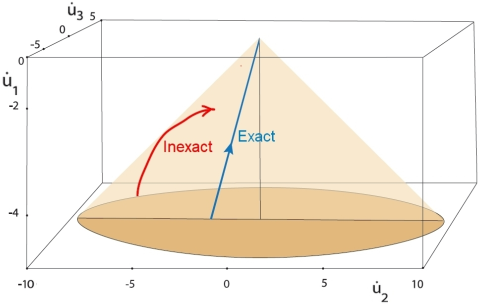

Figure: The evolution of the exact and an inexact solutions in the lower half-cone (shrinking universe). The arrow shows the direction of time; it is always upward.

- Attaining the boundary of the cone (defined by $E_k = 0$) requires infinite time: It is obvious for the exact solution (where $t \to \infty$ corresponds to the apex).

  On approaching a point on the lateral surface, say $\dot{u}_1 \to C$, we have $u_1 \propto -C\,t$. On the other hand, $E_k \to 0, \implies$ all three exponents in E_p tend to zero, $\implies$ $u_1 \to -\infty$. Hence $t \to \infty$.

- The converse: If a limit $t \to \infty$ exists, it is attained at the boundary of the cone, $E_k = 0$, only.

  The limit requires $\lim_{t\to\infty} \ddot{u}_i = 0, \; i = 1, 2, 3$, which infers vanishing of all three exponentials in the Lagrangian. On the other hand, $E_k \neq 0$ together with $H = 0$ implies $E_p \neq 0$, which requires at least one nonzero exponential.

  However:

- $\dot{u}_1$ must have a limit as an increasing function bounded from above. The other two, $\dot{u}_2$, $\dot{u}_3$, may have it or not.

Asymptotics in the neighbourhood of the lateral surface

- While approaching the lateral surface of the cone $E_k = 0$:
  - If the limits, at $t \to \infty$, of $\dot{u}_1,\ \dot{u}_2,\ \dot{u}_3$ are $-g_1,\ -g_2,\ -g_3$ respectively, ($g_1 > 0,\ g_2 - 3g_3 > 0$ and $g_2 + 3g_3 > 0$), then $u_1,\ u_2,\ u_3$ are asymptotically linear functions of time: $u_1 \sim -g_1 t + h_1, \quad u_2 \sim -g_2 t + h_2, \quad u_3 \sim -g_3 t + h_3$.
  - Since the total Hamiltonian $\mathcal{H} = 0 \implies$ each component of potential energy $E_p = -e^{2(u_1 - u_2 - u_3)} - e^{u_2 - 3u_3} - e^{u_2 + 3u_3}$ tends to zero, whence the exponents tend to $-\infty$.
  - Hence, the remainder of E_k exponentially falls with time.
  - Vanishing of E_k on the cone surface requires
    $$3g_1^2 - g_2^2 - 3g_3^2 = 0.$$
  - The above conditions in terms of the scale factors a, b, c mean that $a \sim \exp(2p_a t)$, $b \sim \exp(2p_b t)$, $c \sim \exp(2p_c t)$ with
    $$p_a p_b + p_b p_c + p_c p_a = 0.$$

Kasner-like solutions (1)

- The condition $a \sim \exp(2p_a t)$, $b \sim \exp(2p_b t)$, $c \sim \exp(2p_c t)$, with $p_a p_b + p_b p_c + p_c p_a = 0$, makes these solutions similar to the Kasner solutions of Einstein equations, where the metric reads *[M.P. Ryan Jr. 1972, Ann. Phys.* **70***, 301]*

$$ds^2 = dt^2 - t^{2p_1} dx_1^2 - t^{2p_2} dx_2^2 - t^{2p_3} dx_3^2,$$

  with $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1 \qquad (1)$.
  It is only the matter of time rescaling.

- These constraints on the exponents $p_i, \ i = 1, 2, 3$, infer exactly $\qquad p_1 p_2 + p_2 p_3 + p_3 p_1 = 0$.

- Conversely, by rescaling the time and space scales according to the Lie symmetry, we can get both eqs. (1) for $p_a, \ p_b, \ p_c$ instead of $p_1, \ p_2, \ p_3$. Thus our solutions which end on the lateral cone surface are Kasner-like solutions.

Kasner-like solutions (2)

- The relation between p_a, p_b, p_c shows that at least one and at most two of these exponents are negative.

  Hence

  The universe is collapsing to 0 in one or two dimensions while stretching to ∞ in the other two or one.

- This way, our exact solution is the only solution describing the collapse to zero in all dimensions.

- And this only fully collapsing solution is unstable!

Stability of the Kasner-like solutions

- The asymptotics $t \to \infty$ may be obtained by time rescaling $t \to \sqrt{\varepsilon} t$ ("normal" new time corresponds to high values of the old one).

- Then in the zero order in ε, we regain $u_1 = -g_1 t + h_1, \;\; u_2 = -g_2 t + h_2 \;\; u_3 = -g_3 t + h_3.$

- The next order terms in ε fall exponentially with time, as $e^{-q_1 t}, \; e^{-q_2 t}, \; e^{-q_3 t}$, respectively, with exponents $q_1 = -2(g_1 - g_2 - g_3)t, \; q_2 = -(g_2 - 3g_3)t$, and $q_3 = -(g_2 + 3g_3)t$, respectively. All of them are negative $\Longrightarrow$ the asymptotic solutions are stable.

- When the trajectory reaches the lateral surface, an exchange of the roles of a, b, c and the exponents $p_a, \; p_b, \; p_c$ takes place (see the simulations below).

A couple of simulations of the Kasner-like solutions (1)

The simulations require variables which are of similar order for large time. These are $u := a^2$, $v := b/a$ and $w := c/b$.

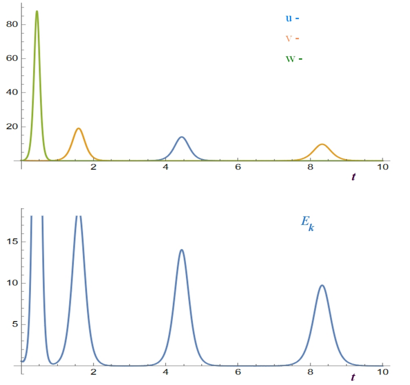

Figure: Dynamics of $u(t)$, $v(t)$ and $w(t)$ for typical initial data, $u(0) = 0.225$, $v(0) = 0.25$, $w(0) = 0.1$, $\dot{u}(0) = -9/4$, $\dot{v}(0) = -5/2$. When all of u, v, w turn to zero, the "kinetic energy" also turns to zero, according to the constraint.

A couple of simulations (2)

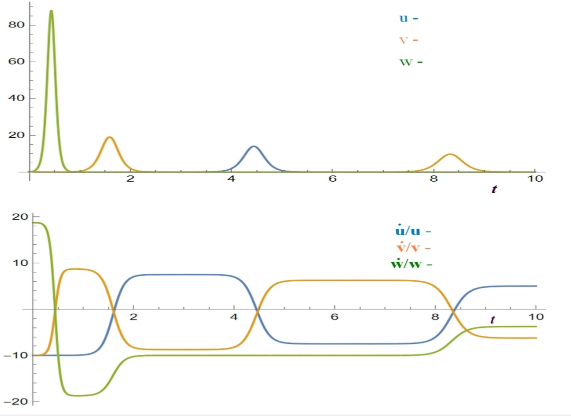

Figure: Dynamics of $u(t)$, $v(t)$ and $w(t)$ and their logarithmic time derivatives for the same initial data. Occasionally all of u, v, w turn to zero, as well as their time derivatives, but in no case do all logarithmic derivatives $\dot{u}/u$, $\dot{v}/v$, $\dot{w}/w$ vanish. This means that the trajectory is incident on the lateral surface of the cone. Note the exchange of the roles of these variables, typical for Kasner's solutions.

Limit and no-limit solutions

- $\dot{u}_1$ must have a limit, say g_1, as an increasing function bounded from above. The other two, $\dot{u}_2$, $\dot{u}_3$, may have it or not.
- If they do not have a limit, then their dynamics for large t takes place within an ellipsoidal horizontal cross section of the cone, $\dot{u}_2^2 + 3\dot{u}_3^2 < g_1^2$.
- Apart from reaching the surface of the cone, returns of the trajectory of $\dot{u}_3$ are possible while u_2 may bounce only once from the wall of the potential.

Solutions with no limit

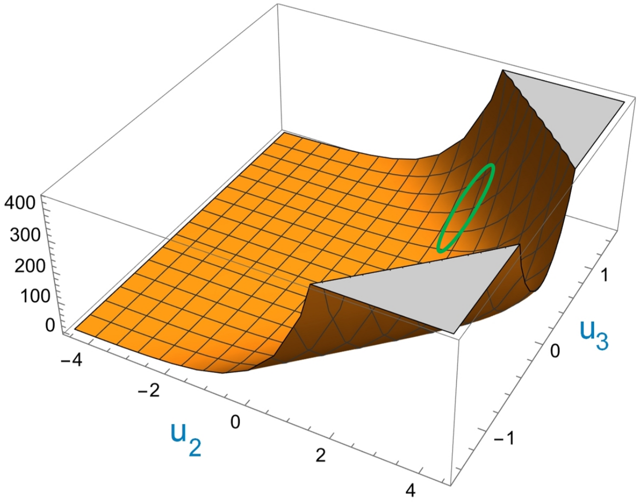

Figure: The potential well when u_1 has reached its limit g_1. The dependent variable $u_3(t)$ may vary periodically. Here the path (in green) has been stretched in the u_2 direction for clarity

Which behaviour is generic?

- The question of genericness may be addressed by means of the Liouville conservation of a cell volume in the 6d phase space.
- We will show how this affects the 3-dimensional volume in the cone of kinetic energy.

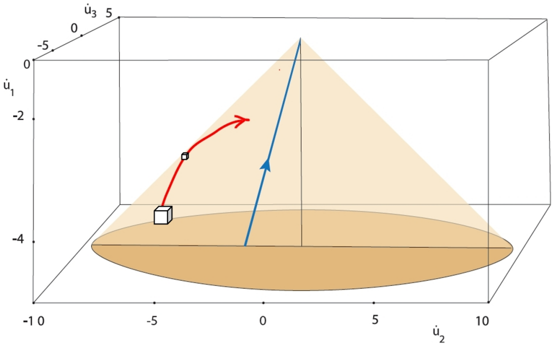

Figure: Dynamics of the phase volume within the cone. The volume shrinks, but proportions in the space of momenta are conserved.

From 6-dimensional phase volume of initial conditions to volume in the cone

- We can make the reduction to a 5D space by replacing one of the variables, say u_3 by the Hamiltonian.
- The distance in the H direction is invariant in time. Hence, the 5D volume will also be invariant in these variables.
- This change of variables has a well-defined Jacobian $J = -3e^{u_2(0)} \sinh\, u_3(0)$.
- This Jacobian is independent of momenta. It means that the (initial or final) phase volume is proportional to the 3D volume in the space of momenta. This in turn infers the genericness of ending on the surface rather than on the vertex of the cone. Hence, ending on the surface is the typical (generic) situation.

To be honest

- Definitely, the lateral surface of the cone is of higher dimension than its apex and thus the solutions which end on the surface are typical, rather than our exact solution.

- It seems, although it is a conjecture rather than a proven fact, that the no-limit solutions are “more typical” than those stable solutions which end on the cone surface. This would explain the chaotic behaviour observed in most simulations.

Conclusions

Summary of results about description through quadrics (1)

- Trajectories of velocities or momenta in the cone of kinetic energy may completely describe the system.
- Although the apparent form of the equation does not provide the trajectories (unlike phase diagrams of one-variable ODE), it provides their asymptotic form.
- It also may provide the direction of evolution and information, whether the boundary values are attained in a finite or infinite time.
- For a given set of initial conditions, the proportion of 3D volumes in the cone is the same as the proportion of phase volumes.

Summary of results about description through quadrics (2)

- These methods work well for Hamiltonian systems whose kinetic part is a quadratic form with constant coefficients.
- It may also work for some space-dependent kinetic parts, if the deformation do not affect the topology.
- In our case, the quadric was single and a cone. If no constraints are imposed on the Hamiltonian, we have a quadric for each value of $\mathcal{H}$ separately. If the kinetic energy is positive definite, the quadrics are ellipsoids.

Summary of results on the BKL scenario (1)

- The BKL system of equations with the constraint $\mathcal{H} = 0$ has exactly one exact solution, up to a time shift.
- The exact solution is the only one which leads to the collapse in all directions and the only one ending in the apex of the cone. As found previously, the asymptotics of the scale factor undergo a power-like collapse.
- The exact solution is unstable to small perturbations, which develop into oscillations with two characteristic incommensurable frequencies (previous result); this leads to chaotic behaviour.
- Reaching the surface of the cone (whether lateral or apex) requires infinite time and vice versa: if exists a limit at $t \to \infty$, the system inevitably ends on the surface (including the apex).

Summary of results on the BKL scenario (2)

- The generic solutions are those which end on the lateral surface of the cone, rather than its apex).
- Contrary to the exact solution, in the asymptotics of the non-exact solutions, the spatial scale factors are exponential functions of time.
- These solutions describe the universe collapsing in one or two directions while stretching in the other two or one, like Kasner's solutions.
- Also, contrary to the exact solution, these solutions are stable to small perturbations.
- Still more frequent may be solutions without limits (conjecture). This leads to chaos again.

Thank you for your attention!