Predicting Nonabelian Anyons

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Our early theoretical predictions of anyon statistics, their properties, and nonabelian anyons, were accomplished together with Ralph Menikoff at Los Alamos National Laboratory.

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A brief overview of the talk

1. Anyons and nonabelian anyons
2. A universal kinematical group for quantum mechanics
3. Diffeomorphism groups and their unitary representations
4. Predicting anyons and nonabelian anyons: Representations induced from the braid group
5. Inequivalent representations: A vast universe of quantum systems
6. Remarks on an issue of scientific and journalistic integrity
7. Relation to Heisenberg quantization (if time permits)
8. Research directions and open questions (if time permits)

1. Anyons and nonabelian anyons

We are all familiar with **bosons** and **fermions**. Their exchange statistics for N indistinguishable particles are governed by characters (one-dimensional unitary representations) of the symmetric group S_N.

Parastatistics refers to hypothetical particles whose exchanges are governed by higher-dimensional representations of S_N.

Exotic possibilities in two space dimensions

"**Anyons**" are quantum particles or excitations in two-space with exchange statistics' intermediate between bosons and fermions ($\exp i\theta$). They are associated with surface phenomena in the presence of magnetic flux. Exchanges via paths in configuration space are governed by characters of the braid group B_N.

The exchange statistics of **nonabelian anyons** satisfy unitary representations of B_N of dimension greater than one — i.e., nonabelian representations.

Predictions and experiments

The ideas behind **bosons** and **fermions**, named for Bose and Fermi respectively, were developed in the 1920s.

The idea of **parastatistics** was introduced by Green (1953) and developed further by Messiah & Greenberg (1964).

The intermediate statistics of **anyons** was first predicted by Leinaas & Myrheim (1977), independently by Goldin, Menikoff, & Sharp (1981), and for the third time by Wilczek (1982). To my knowledge the braid group was first explicitly identified as governing their exchange statistics in our immediately subsequent publications (1983).

Direct experimental confirmation followed decades later, by Bartolomei et al. (2020) and Nakamura et al. (2020).

Nonabelian anyons were first predicted by Goldin, Menikoff, and Sharp (1985). A concrete physical realization has been constructed by Andersen et al. (2023).

2. A universal kinematical group for quantum mechanics

In this talk I explore the foundations of our predictions, which I think have not-yet-explored implications for quantum theory.

My main point is to highlight how and why the "local current algebra" of mass and flux densities in **physical space** M generates directly a **universal kinematical group** G for quantum physics, and to say what that means.

Understanding the unitary representations of G led to our prediction of intermediate anyon statistics in two-space, and then to our prediction of nonabelian anyons. We also reached a clear understanding of how topology enters into quantum kinematics, and resolved issues having to do with "multivalued wave functions" in quantum mechanics.

This local current algebra was introduced by Dashen & Sharp (1968) in the context of nonrelativistic quantum field theory. Here I take a different perspective, one that explains its universality.

The Lie algebra of mass and flux densities in physical space M

Consider any system having mass in a physical space M. Typically $M = \mathrm{R}^3$. But we may take $M = \mathrm{R}^2$, or a general manifold (e.g., with nontrivial topology), or a manifold with boundary, or even a manifold with singularities.

At a fixed (but arbitrary) time, we can take measurements of the ***mass density***. As we may have point paricles, extended massive objects, or both, the mass density should be described by a distribution, rather than necessarily a smooth function.

The test-function space for such a distribution is a space of C^∞ real-valued functions f on M. **Local measurements** suggest taking functions with compact (but arbitrary) support. Mathematically, we endow this function space $\mathcal{D}$ with the topology of uniform convergence in all derivatives.

The Lie algebra of mass and flux densities in M (continued)

Alternatively we can envision measuring, at the fixed (but arbitrary) time, the mass density flux – i.e., the ***momentum density*** in M. This must likewise be described as a distribution, but vector-valued.

The test-function space for the momentum density will consist of C^∞ (tangent) vector fields g on M, with compact (but arbitrary) support. Again, we may endow this space with the topology of uniform convergence in all derivatives.

The geometry of M provides us with the **Lie bracket** of vector fields $[\mathrm{g}_1, \mathrm{g}_2]$. Vector fields g act on scalar functions f via the Lie derivative $L_{\mathrm{g}} f$. Physically, this is the statement that in the right units, g tests for flux in the quantity that f test for: i.e., **momentum density refers to the flux of mass**.

If M has a boundary ∂M, we naturally require test-vector fields to be tangential at the boundary. If M has singularities, we naturally require the vector fields to vanish there.

The Lie algebra of mass and flux densities in M (continued)

To describe a general quantum kinematics, we represent mass and momentum density distributions (respectively) by self-adjoint operators $\hat{\rho}(f)$ and $\hat{J}(\mathrm{g})$ in Hilbert space. We thus obtain a **Lie algebra** of densiies and currents: the natural semidirect sum of the scalar functions on M with the tangent vector fields on M:

$$[\hat{\rho}(f_1), \hat{\rho}(f_2)] = 0\,, \quad [\hat{\rho}(f), \hat{J}(\mathrm{g})] = i\hbar\hat{\rho}(L_{\mathrm{g}} f), \tag{2.1}$$

$$[\hat{J}(\mathrm{g}_1), \hat{J}(\mathrm{g}_2)] = -i\hbar\hat{J}([\mathrm{g}_1, \mathrm{g}_2])\,, \tag{2.2}$$

Planck's constant enters as a structure constant for this Lie algebra.

Parallel remarks pertain to electric charge and electric current densities, and to other physical quantities that may characterize a physical system.

A universal group describing quantum kinematics in M

When represented by self-adjoint operators, the above Lie algebra is just the **local current algebra** proposed (in a singular form) by Dashen and Sharp (1968) and regularized by GG (1971), obtained from second-quantized canonical fields. But the basis for it suggested here is more direct and fundamental.

The corresponding group G, obtained by exponentiating these commutation relations, is represented by unitary operators in $\mathcal{H}$. It is the natural semidirect product of the space of smooth, compactly-supported scalar functions f under addtiion, with a group $\mathcal{K}$ of compactly-supported C^∞ diffeomorphisms $\phi : M \to M$ under composition:

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + \phi_1 f_2, \phi_1 \phi_2), \tag{2.3}$$

where $\phi_1 f_2 = f_2 \circ \phi_1$, and $\phi_1 \phi_2 = \phi_2 \circ \phi_1$.

This group $\mathcal{D} \rtimes \mathcal{K}$ is the **universal kinematical group** in the title of this section, whose representations led to our predictions of anyon and nonabelian anyon statistics.

A universal kinematical group for quantum kinematics (continued)

Just as a uniform vector field on $\mathbb{R}^d$ generates a 1-parameter group of translations, a compactly supported vector field g on M generates a 1-parameter group of diffeomorphisms of the spatial manifold – a ***flow*** $\phi_s : M \to M$ given by the PDE,

$$\partial_s \phi_s(x) = \mathrm{g}(\phi_s(x)) \text{ with } \phi_{s=0}(x) \equiv x, \tag{2.4}$$

for $x \in M$ and $s \in \mathbb{R}$. Label such a flow ϕ_s^{g} after the vector field g that generates it. Then we may write a unitary representation of Eq. 2.3 as $U(f)V(\phi)$, where $U(f) = \exp i\hat{\rho}(f)$ and $V(\phi^{\mathrm{g}}) = \exp i\hat{J}(\mathrm{g})$. The self-adjoint local currents (i.e., the observables) are recovered easily in such a representation.

All this just expresses mathematically the statement that **momentum flux entails the transfer of mass** (resp., **electric current density entails the transfer of charge**).

Note that ***physical space*** M is not ***configuration space***. ***Momentum density in physical space*** is not ***momentum in phase space***. These are very different constructs.

3. Diffeomorphism groups and their unitary representations

To sum up, a continuous, irreducible unitary representation $U(f)V(\phi)$ of $G = \mathcal{D} \rtimes \mathcal{K}$ in Hilbert space $\mathcal{H}$ describes a quantum system in physical space M. All possible quantum kinematics of systems with mass, charge, or other scalar density are thus described!

For any compact region $B \subset M$, a subgroup of $\mathcal{K}$ is defined by restricting the support of the diffeomorphisms in the subgroup to B. This establishes a localization structure in $\mathcal{K}$, as well as in $\mathcal{D}$. Thus does G express a kind of **local kinematical symmetry**. We endow both $\mathcal{D}$ and $\mathcal{K}$ with the topology of uniform convergence in all derivatives.

The distinct (inequivalent) representations of $\mathcal{D} \rtimes \mathcal{K}$ now yield different quantum systems. This leads to a unification where ***the class of unitary representations of a single group predicts the kinematics of all possible quantum mass configurations, exchange statistics, and more***.

So we need to have a look at how to classify the unitary repreentations of diffeomorphism groups leading to this universe of possibilities.

General framework for the representation theory

I shall focus here on representations of the semidirect product. Under very general conditions, a unitary representation of $G = \mathcal{D}(\mathcal{M}) \rtimes Diff_0(M)$ may be written

$$[U(f)\Psi](\gamma) = \exp i\langle \gamma, f \rangle \, \Psi(\gamma)\,, \tag{3.1}$$

$$[V(\phi)\Psi](\gamma) \,=\, \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}. \tag{3.2}$$

Here M is physical space, but γ belongs to (a configuration space) Δ, a subset of the continuous dual $\mathcal{D}'$ (the space of distributions on M). $\mathcal{D}'$ carries the dual action of $\mathcal{K}$, for which Δ is an invariant set.

For example, with $M = \mathbb{R}^3$, Δ may consist of sums of Dirac δ-distributions (evaluation functionals) on $\mathcal{D}$; i.e., $\gamma = \delta_{x_1} + \cdots + \delta_{x_N}$ where $\langle \delta_x, f \rangle = f(x)$. A diffeomorphism ϕ acts on γ by taking δ_{x_j} to $\delta_{\phi(x_j)}$ – effectively, we have N-point subsets of $\mathbb{R}^3$. This representation describes N indistinguishable quantum particles.

General framework for the representation theory (continued)

$$[U(f)\Psi](\gamma) = \exp i\langle\gamma, f\rangle\,\Psi(\gamma)\,,$$

$$[V(\phi)\Psi](\gamma) \;=\; \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}.$$

To continue, μ is a measure on Δ quasi-invariant under the action of G (i.e., the class of measure zero sets is preserved). For example, μ can be a local Lebesgue measure $dx_1 ... dx_N$ on the N-particle configuration space, or it can be a Poisson or Gibbs measure in an infinite configuration space.

Ψ is a function on Δ with values in a complex Hilbert space $\mathcal{W}$. Thus Ψ may be complex-valued (i.e., $\mathcal{W} = \mathbb{C}$), or it may have multiple components (e.g., $\mathcal{W} = \mathbb{C}^n$). $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathcal{W}$. $\langle\Psi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}$ is integrable with respect to μ.

χ is a ***measurable unitary 1-cocycle*** acting in $\mathcal{W}$ (described next).

Inner product in $\mathcal{H}$, and the cocycle equation

$$[U(f)\Psi](\gamma) = \exp i\langle\gamma, f\rangle\, \Psi(\gamma)\,,$$

$$[V(\phi)\Psi](\gamma) \;=\; \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)}.$$

The inner product in $\mathcal{H}$ is given by

$$(\Phi, \Psi) = \int_\Delta \langle\Phi(\gamma), \Psi(\gamma)\rangle_{\mathcal{W}}\, d\mu(\gamma)\,, \tag{3.3}$$

where $\langle\;,\;\rangle_{\mathcal{W}}$ is the inner product in $\mathcal{W}$. Thus Ψ belongs to $\mathcal{H} = \mathcal{L}^2_{d\mu}(\Delta, \mathcal{W})$.

The ***cocycle equation*** for χ is,

$$\chi_{\phi_1\phi_2}(\gamma) \;=\; \chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma)\,. \tag{3.4}$$

The above equations all hold outside sets of μ-measure zero in Δ.

Cocycles and induced representations

Define the ***stability subgroup*** $K_\gamma \subset \mathcal{K}$ (γ fixed) to consist of the diffeomorphisms $\phi \in \mathcal{K}$ for which $\phi\gamma = \gamma$.

Note that with $\phi_1, \phi_2 \in K_\gamma$, Eq. (3.4) is a unitary representation of K_γ in $\mathcal{W}$. When $\mathcal{W} = \mathbb{C}$, it is a ***character*** of K_γ.

When we know Δ and the action of G on Δ, one way to obtain cocycles in Eqs. (3.2)-(3.4), when certain conditions are satisfied, is by ***inducing*** from unitary representations of K_γ.

This method of induced representations, generalizing George Mackey's seminal work, realizes the Hilbert space via ***equivariant wave functions*** $\tilde{\Psi}$ ***on a covering space or fiber bundle over*** Δ to which $\mathcal{K}$ lifts naturally. The equivariance is with respect to the representation of the stability subgroup.

Cocycles and induced representations (continued)

For a choice of Δ with quasiinvariant measure μ, one always has $\mathcal{W} = \mathbb{C}$ and $\chi_\phi(\gamma) \equiv 1$ characterizing a unitary group representation on complex-valued wave functions (with a trivial cocycle). This cocycle is induced by the character of K_γ that is identically 1.

But inequivalent (non-cohomologous) unitary 1-cocycles describe unitarily inequivalent representations of G. And inducing from inequivalent unitary representations of the stability subgroup yields inequivalent cocycles. This is ***the fundamental reason for exchange statistics of indistinguishable particles, and for anyon statistics in two-space***.

We remark that the system of Radon-Nikodym derivatives $[d\mu_\phi/d\mu](\gamma)$ is also a real 1-cocycle, and consequently so is its square root.

4. Predicting anyons and nonabelian anyons: The fundamental group of configuration space

Next we see ***how the stability subgroup*** K_γ of a configuration $\gamma \in \Delta$ ***maps to the fundamental group*** (first homotopy group) of Δ.

Consider the configuration space $\Delta^{(N)}$ of N-point configurations in 3-space. A diffeomorphism that leaves fixed $\gamma = \{x_1, ..., x_N\} \subset \mathrm{R}^3$ (or equivalently, $\gamma = \delta_{x_1} + \cdots + \delta x_N$ in the continuous dual of $\mathcal{D}$) can do so by leaving each point fixed; but also, by permuting the points.

So here K_γ maps via a natural surjective homomorphism onto the **symmetric group** S_N. Unitary representations of S_N provide continuous unitary representations of K_γ. These in turn induce representations of the full diffeomorphism group.

The fundamental group of configuration space (continued)

When Ψ is complex-valued and $\chi \equiv 1$, we have **bosonic** exchange symmetry. This corresponds to inducing by the **identity representation** of S_N. The **alternating representation** of S_N, $N \geq 2$, gives **fermionic** exchange symmetry.

The **higher-dimensional representations** of S_N induce representations that describe particles satisfying the **parastatistics** of Greenberg and Messiah. Here we have wave functions taking vector values, according to the dimension of the representation of S_N.

Thus unitary representations of G describe the kinematics of particles satisfying all of these possibilities.

For $d \geq 3$, S_N is the fundamental group of the configuration space $\Delta^{(N)}$. This is not a coincidence. We shall see why it is always necessary to consider the topology of Δ.

The fundamental group of configuration space (continued)

The fundamental group (first homotopy group) is defined from homotopically equivalent classes of closed paths (i.e., pointed loops). But diffeomorphisms of the physical space act **not only on objects, but on paths** in configuration space. And a path in G from the identity to K_γ generates a pointed loop based at γ.

Thus the stability subgroup **maps naturally to the fundamental group**, even in contexts where the fundamental group is different from S_N.

Let us remark further that when the configuration space is a space of **embeddings** of a compact manifold N into physical space M, then when γ is such an embedding the stability subgroup $K_\gamma \subset Diff_0(M)$ maps naturally onto $Diff(N)$, whose unitary representations in turn induce representations of $Diff_0(M)$.

Predicting anyons and nonabelian anyons in two-space

When $d = 2$ and $N \geq 2$, the fundamental group of the N-particle configuration space $\Delta^{(N)}$ is no longer S_N, but the **braid group** B_N.

The 1-dimensional unitary representations (characters) of B_N then induce representations of the local current algebra describing **intermediate** (anyon) exchange statistics: a single counterclockwise exchange of two particles introduces an arbitrary phase $\exp i\theta$.

Because self-adjoint operators describing relative angular momentum can be obtained from the current algebra representation, the shifted angular momentum spectrum is also found in this framework. It describes fully the kinematics of anyons.

We also see clearly that **distinguishable** particles in two-space can pick up intermediate phases when one fully circles another. Here the fundamental group is the group of **colored braids**.

Predicting anyons and nonabelian anyons (continued)

But **higher-dimensional unitary representations of B_N** exist. They serve perfectly well to induce unitary representations of the kinematical group G by the above construction. These describe **nonabelian anyons** in two space dimensions. Nonabelian anyons have also been termed **plektons**.

The role of nonabelian repersentations of B_N in two-space is in close analogy with the induced representations governed by S_N that describe parastatistics in three dimensions.

From the perspective of unitary group representations, the predicted kinematics of nonabelian anyons appears obvious. From other perspectives that were current in the early 1980s, it was not. It is interesting to consider historically why not, but that is beyond today's talk.

Predicting anyons and nonabelian anyons (continued)

Our prediction of anyons was one of three independent discoveries: Leinaas & Myrheim (1977), ours at Los Alamos (1980-81), and Wilczek (1982) who coined the term "anyons". Our group (1983) first identified the braid group as governing anyon exchange statistics.

The theoretical possibility of **nonabelian anyons**, also called **plektons**, was established and first pointed out by our group (1985).

As is now well-known, these ideas find application in condensed matter physics, in the theory of the quantum Hall effect, in the theory of quantum vortices, and most recently in quantum computing. Direct experimental confirmation of anyonic excitations in surface phenomena occurred in 2020, over 40 years after the first predictions!

Next I shall briefly survey the vast array of quantum kinematical systems described by unitary representations of G.

5. Inequivalent representations: The vast universe of quantum systems

The irreducible unitary representations of G embrace all known quantum-mechanical possibilities – and have predicted previously unsuspected ones now confirmed. This gives us confidence in the importance of our group-theoretic approach to quantum mechanics and quantum field theory.

The generality of this description allows for its application to non-simply-connected manifolds, manifolds with boundary, and manifolds with singular points.

In this quantum kinematics, configuration space and classical phase space are recovered separately in each representation. The resulting theory may then be regarded as a "quantization," relating closely to the excellent talk by Jerzy Kijowski on Tuesday.

In each system a natural constraint is the ***continuity equation*** for the time-evolution of operator-valued distributions: $\partial_t \hat{\rho} = -\nabla \cdot \hat{J}$. It is a mass conservation constraint imposed by the kinematics on the dynamics, and thus on the (not yet specified) Hamiltonian.

The universe of quantum systems

(a) ***N*-particle** quantum mechanics in M ($N = 1, 2, 3, ...$), with particles distinguished by their masses (or charges), or (next case) indistinguishable.

(b) Systems of indistinguishable particles obeying **Bose or Fermi exchange statistics** in two or more space dimensions. In one space dimension, it is interesting (and made clear) that the exchange statistics is of dynamical not kinematical origin.

(c) Systems of indistinguishable particles (or excitations) obeying intermediate, **anyon statistics** in two space dimensions (for any given anyonic phase shift under counterclockwise exchange): ***an early, independent prediction with these methods***

(d) Systems of **distinguishable anyonic particles in two-space**, with distinct relative phase shifts under full relative counterclockwise rotation: ***first pointed out with these methods***

(e) Systems of particles obeying **parastatistics** (in two or more space dimensions)

(continued)

The universe of quantum systems (continued)

(f) Systems of **nonabelian anyons** in two-space: ***first predicted with these methods***

(g) Systems of **tightly bound charged particles** – **point dipoles, quadrupoles, etc.**

(h) Particles with spin, arranged in **spin towers** according to representations of the general linear group

(i) Particles with **fractional spin**, in two space dimensions: ***described rigorously with these methods***

(j) Systems of **infinitely many particles**, in locally finite configurations, corresponding to a **free or interacting Bose gas** (Poisson measures, respectively Gibbs measures)

(k) Systems of **infinitely many particles obeying Fermi statistics, or exotic statistics**

(l) Systems of **infinitely many particles with accumulation points** in fractal configurations: ***suggested rigorously by these methods***

(m) **Quantized vortex systems**: vortex filaments in two space dimensions, or ribbons in three-space, from representations of area- (resp., volume-)preserving diffeomorphisms

(continued)

The universe of quantum systems (continued)

(n) Configurations of **extended objects**, including **loops and strings, knotted configurations, and objects with nontrivial topology and/or nontrivial internal symmetry**

(o) Quantum mechanics on physical spaces that themselves are **manifolds with boundary, with singularities, or with nontrivial topology**

(p) Quantum particles where the (kinematical) continuity equation requires **nonlinear time-evolutions** (some previously studied), possibly equivalent to linear theories via nonlinear gauge transformations, but otherwise violating the "no signal" property: ***first predicted with these methods***

(q) Other ... ? There may well be interesting possibilities still undiscovered.

The mathematical theory behind many of the above descriptions is incomplete or only partially developed. There are many unanswered questions, opportunities for new constructions, and possibly new predictions to be made of a fundamental nature.

6. Remarks on an issue of scientific and journalistic integrity

The next rew slides are unfortunate. But they have proved essential for me to include when I speak on this subject.

A long series of major articles and reviews, beginning in 1982 with Wilczek's papers, have included ***no citation or insignificant citation*** of our publications and early predictions. Some flagrant omissions have been intentiona. On being informed, authors sometimes promise to cite our work in the future – if they respond at all.

Continuing this pattern recent experimental articles, drawing considerable attention from the scientific community to anyons and nonabelian anyons, include ***no citation*** of our many publications and early predictions. Some authors do not answer cordial correspondence. Others promise to cite our work in the future.

It is a rather strange experience to be "written out of history" before one's eyes over four decades – despite extensive documentation, the absence of any dispute, and no disagreement with the correctness of our theoretical development. There is considerable "opportunity cost" to an active theorist.

Scientific and journalistic integrity (continued)

The saga continues. Several recent, laudatory accounts in the popular scientific literature (e.g. *Discover Magazine* and *Quanta Magazine*) erroneously attribute the prediction of anyons exclusively to Wilczek, with ***no mention*** of their prior prediction by anyone else.

Editors have to this point disregarded corresponcence or ***dishonestly*** refused to post corrections, violating well-established standards of scientific and journalistic ethics.

Wikipedia still has no reference to our work in the entry about anyons. An editor at Wikipedia with the screen name "HouseofChange" has ***expunged*** repeated efforts by researchers to include the undisputedly correct attributions, making false and malicious allegations against those attempting to include the citations.

This level of dishonesty has no place in science. It is not a small issue, as it affects my close colleagues and me. We devoted many years in our early careers to the work that led to our results in the early 1980s (and since then).

Conequences for others and for the field

We have learned through our recent experiences that ***intentional failure to cite is a systemic problem in the world of science.*** It has been and is profoundly harmful to the careers and morale of those less able to speak out than we are.

Those who can be especially hurt include minority scientists, women, researchers situated in developing countries, and young researchers generally, who have much to lose by raising this issue when it affects them.

David Sharp and I are in positions of relative security, having had fulfilling careers with many privileges and opportunities. We have determined to pursue and publicize the issue. It should matter to everyone.

For a more detailed discussion, see "The Prediction of Anyons: Its History and Wider Implications" on the arXiv (December 2022): https://arxiv.org/abs/2212.12632

7. Relation to Heisenberg quantization (if time permits)

Returning to our group-theoretic description, when a configuration space manifold Δ is established via a unitary representation $U(\cdot)V(\cdot)$ of G, the contangent bundle of Δ may be interpreted as providing a classical phase space – for which the unitary representation at hand provides a quantization.

The classical functions on N-particle configuration spaces in d space dimensions that end up "quantized" in this way are those of the form $\Sigma_{j=1}^{N} f(q_j)$ and $\Sigma_{j=1}^{N} g(q_j) \cdot p_j$, where f and the d components of g are smooth, compactly supported functions of the d-component coordinate q_j, and p_j is the d-component vector of the corresponding momentum.

Note that $U(\cdot)V(\cdot)$ is not in general a unique quantization of the phase space! Inequivalent cocycles provide distinct ways to quantize.

Our perspective is that nature does not actually quantize; rather, quantization is our way of addressing the inverse problem of the classical limit to a quantum system.

8. Research directions and open questions (if time permits)

Related directions of research in which I'm presently engaged:

Relativistic and general relativistic generalizations of the diffeomorphism group description of quantum physics (with David Sharp, Alexander Smith, and students Andrea Russo and Shadi Ali Ahmad)

Geometric quantization of extended vortex configurations, and resulting anyonic phase parameters (with Francois Gay-Balmaz and Cornelia Vizman)

Diffeomorphism group representations and cocycles describing quantum multipole and other planar configurations, associated anyonic phase parameters, and how they enter the theory (with Rutgers graduate student Hongyi Shen)

Some additional open questions (as far as I know):

How can we describe a wider, completely general theory of measures quasiinvariant under diffeomorphisms, on various categories of infinite-dimensional configuration spaces?

What can serve as the most general possible quantum configuration space, beyond the space of distributions?

There are many open questions about relativistic quantum field theory of anyons, and the anyon gas under various kinematical and dynamical assumptions.

Can the methods mentioned be extended from the "toy model" of an incompressible fluid obeying Euler's equations, to the compressible fluid obeying the Navier-Stokes equation?

How should we reformulate foundations of quantum mechanics without stronger-than-necessary axioms of linearity? Is a linearizable gauge necessary to the no-signal property? What are the general conditions on time-evolution equations (PDEs) and quantum kinematics necessitated by the absence of communication via quantum correlations?

Other directions ...

Thank you for your attention.

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