On Automorphisms of Complex b^k-Manifolds

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- **b-calculus** (Melrose) is a framework for dealing with singular partial differential operators that degenerate along a hypersurface Z in their domain manifold M. It has applications to index theory.
- Can consider "b versions" of classical geometries.
- **b-symplectic geometry** (aka "**log-symplectic geometry**") has been considered by various authors (Guillemin, Miranda, Pires, ...).
- Mendoza introduced **b-complex geometry**
- One can also consider $\mathbf{b^k}$**-calculus**, $k \geq 2$, where k encodes the order of degeneracy along $Z \subseteq M$. This was pursued by Scott.
- We are interested in understanding **automorphisms** in b and b^k-complex geometry.

Definition

A **b-manifold** is a smooth manifold M together with a closed hypersurface $Z \subseteq M$.

The **b-tangent bundle** is the vector bundle (Lie algebroid, in fact) ${}^bTM$ over M whose smooth sections correspond to the smooth vector fields on M that are tangent to Z.

- Let $M = \mathbb{R}^2$, $Z = \mathbb{R} \times \{0\}$. Then ∂_x and $y\partial_y$ are a frame for ${}^bTM$.
- Can also define b^k tangent bundles ${}^{b^k}TM$ for $k \geq 2$ whose smooth sections are smooth vector fields on M which are "tangent to Z to kth order".
- Latter notion is not coordinate invariant. Additional infinitesimal data needs to be specified along Z to define ${}^{b^k}TM$ (Scott).

Recall that complex structures can be defined without reference to holomorphic charts. The equivalence of the definition below to the one via holomorphic charts is given by the Newlander-Nirenberg theorem.

Definition

A **complex manifold** is an even-dimensional smooth manifold M equipped with a subbundle W of the complexified tangent bundle satisfying:

(i) $\mathbb{C}TM = W \oplus \overline{W}$,

(ii) W is involutive.

complex b-manifolds (introduced by Mendoza) and **complex b^k-manifolds** may be defined in exact analogy, simply by replacing the role of the tangent bundle by the b-tangent bundle (resp. b^k-tangent bundle).

- For simplicity, we work with $M = \mathbb{R}^2$ and $Z = \mathbb{R} \times \{0\}$ throughout.
- Any complex structure on $\mathbb{R}^2$ is spanned by a single complex vector field

$$L = X + iY$$

with X and Y pointwise linearly independent.
- The choice of defining L is unique up to smooth rescaling.
- Same story in b^k case, with the vector fields X, Y being pointwise-independent as sections of ${}^{b^k}TM$.

Theorem (Newlander-Nirenberg in dimension 2)

If $L = X + iY$, where X, Y are pointwise-independent, smooth vector fields on $\mathbb{R}^2$, then, locally near any point, there is a diffeomorphism θ such that

$$\theta_*(L) \sim L_0 := \partial_x + i\partial_y$$

- Here $\sim$ indicates equality after a smooth rescaling.
- "no local invariants in complex geometry"
- integrability unecessary in dimension 2.

Mendoza has shown there are no "formal local invariants" in complex *b*-geometry, but the natural b and b^k analogs of the Newlander-Nirenberg theorem seem to still be lacking. Although, we do have the following:

Theorem (B-F)

Consider $\mathbb{R}^2$ with b-complex structure defined by $L = X + iY$ where X, Y are smooth vector fields tangent to the x-axis that are pointwise-independent as sections of ${}^bTM$. Suppose there exists a smooth function h with $Lh = 0$ such that $dh(0,0) \neq 0$. Then, there exists a local coordinate change at $(0,0)$ such that

$$\theta_*(L) \sim L_1 := \partial_x + iy\partial_y.$$

Conclusion: there are no local invariants for complex *b*-manifolds that admit nontrivial "*b*-holomorphic" functions.

Definition

The **standard complex b^k-structure** of $\mathbb{R}^2$ is spanned by

$$L_k = \partial_x + iy^k\partial_y \qquad k = 0, 1, 2, \ldots$$

An **automorphism** of $(\mathbb{R}^2, L_k)$ is a diffeomorphism θ of $\mathbb{R}^2$ such that $\theta_*(L_k) \sim L_k$.

- Note $L_0 = \partial_x + i\partial_y$, and $\mathrm{Aut}(\mathbb{R}^2, L_0)$ is the usual $\mathrm{Aut}(\mathbb{C})$ (biholomorphic maps).
- By applying an order-2 automorphism $(x, y) \mapsto (\pm x, -y)$, one may restrict attention to automorphisms of $(\mathbb{R}^2, L_k)$ that fix the open upper half space $\mathbb{H}$.

Definition

$$\mathrm{Aut}_+(\mathbb{R}^2, L_k) = \{\theta \in \mathrm{Aut}(\mathbb{R}^2, L_k) : \theta(\mathbb{H}) = \mathbb{H}\}.$$

Theorem (B-F)

$\mathrm{Aut}_+(\mathbb{R}^2, L_1) \cong \mathbb{R} \times \mathbb{R}$. *It is generated by:*

$$\textit{Horizontal translations:} \quad (x, y) \mapsto (x + t, y)$$
$$\textit{Vertical scalings:} \quad (x, y) \mapsto (x, e^t y)$$

Theorem (B-F)

For $k \geq 2$, $\mathrm{Aut}_+(\mathbb{R}^2, L_k) \cong \mathbb{R}_+ \ltimes \mathbb{R}$, *the "*$ax + b$ *group". It is generated by:*

$$\textit{Horizontal translations:} \quad (x, y) \mapsto (x + t, y)$$
$$\textit{Hyperbolic transformations:} \quad (x, y) \mapsto (e^t x, e^{-\frac{t}{k-1}} y)$$

Remark: The action of $\mathrm{Aut}_+(\mathbb{R}^2, L_k)$ on the singular hypersurface $Z = \mathbb{R} \times \{0\}$ is faithful for $k \geq 2$, but not for $k = 1$.

Basic observation: the diffeomorphism

$$\theta : \mathbb{H} \to \mathbb{R}^2 \qquad\qquad \theta(x, y) = (x, \log y)$$

satisfies

$$\theta_*(\partial_x + iy\partial_y) = \partial_x + i\partial_y$$

and defines an isomorphism of complex manifolds

$$\theta : (\mathbb{H}, L_1) \to (\mathbb{R}^2, L_0).$$

Similarly, for $k \geq 2$,

$$\phi : \mathbb{H} \to \mathbb{H} \qquad\qquad \phi(x, y) = (-x, \frac{1}{(k-1)y^{k-1}})$$

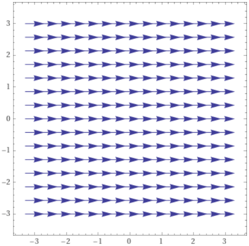
satisfies

$$\phi_*(\partial_x + iy^k\partial_y) = \partial_x + i\partial_y$$

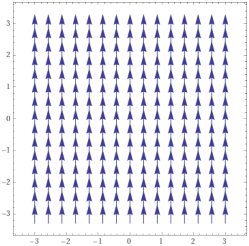
and defines an isomorphism of complex manifolds

$$\phi : (\mathbb{H}, L_k) \to (\mathbb{H}, L_0).$$

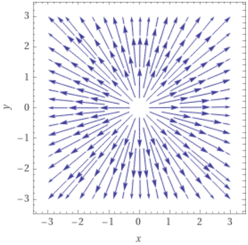
Know the automorphisms of $(\mathbb{R}^2, L_0)$. It's $\mathrm{Affine}(\mathbb{C})$



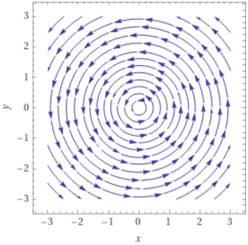
∂_x



∂_x

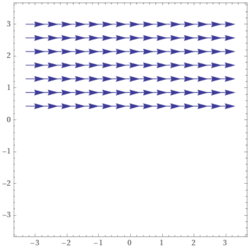


$x\partial_x + y\partial_y$

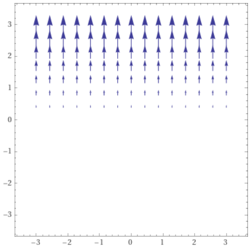


$-y\partial_x + x\partial_y$

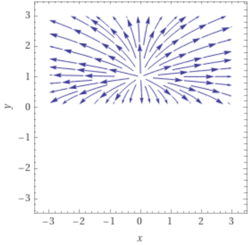
Pull back by $(x, y) \mapsto (x, \log y) : (\mathbb{H}, L_1) \overset{\cong}{\to} (\mathbb{R}^2, L_0)$



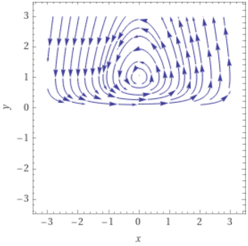
∂_x



∂_y

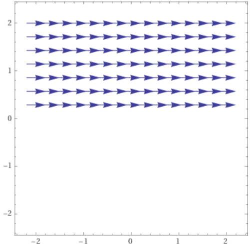


$x\partial_x + y\log(y)\partial_y$

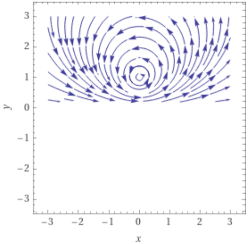


$-\log(y)\partial_x + xy\partial_y$

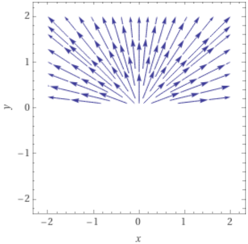
Know the automorphisms of $(\mathbb{H}, L_0)$. It's $PSL(2, \mathbb{R})$.



∂_x

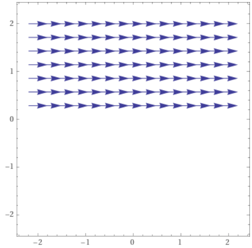


$(x^2 - y^2 + 1)\partial_x + 2xy\partial_y$

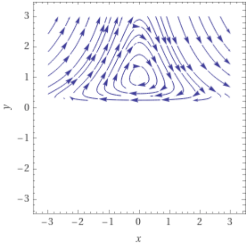


$x\partial_x + y\partial_y$

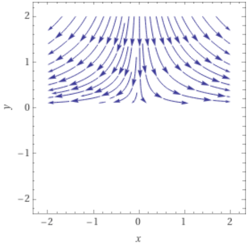
Pull back by $(x, y) \mapsto (-x, \frac{1}{(k-1)y^{k-1}}) : (\mathbb{H}, L_k) \xrightarrow{\cong} (\mathbb{H}, L_0)$



∂_x



$(x^2 - \frac{1}{y^2} + 1)\partial_x - 2xy\partial_y$



$x\partial_x - \frac{1}{k-1} y \partial_y$

Consider the strip $\Omega = \{(x, y) : -\pi < x < 0\}$

Lemma

$h(x, y) = ye^{-ix}$ defines an isomorphism $(\Omega \cap \mathbb{H}, L_1) \to (\mathbb{H}, L_0)$.

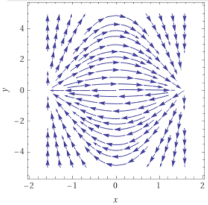
Using a similar "pullback and extend method", we find:

Theorem (B-F)

The vector field

$$y \cos x \partial_x - y^2 \sin x \partial_y$$

defines a 1-parameter group in $\mathrm{Aut}_+(\Omega, L_1)$ that doesn't extend to $\mathrm{Aut}_+(\mathbb{R}^2, L_1)$.



It is interesting to consider b analogues of classical spaces of holomorphic functions such as:

- Segal-Bargmann space: a Hilbert space of entire functions satisfying $f(z) = \int_{\mathbb{C}} |f(z)|^2 e^{-|z|^2}\, dz$
- (Weighted) Bergman space
- Hardy space

One can pose a variety of questions about infinite vs finite-dimensionality these functions spaces.

Example

The function $h(x, y) = ye^{-ix}$ on $\mathbb{C}$ is an example of an entire "*b*-holomorphic function", i.e. $L_1 h = 0$ for $L_1 = \partial_x + iy\partial_y$. Its restriction to $(\mathbb{H}, L_1) \cong (\mathbb{R}^2, L_0)$ belongs to the classical Segal-Bargmann space. Considering its powers, one concludes that "*b*-Segal-Bargmann space" is infinite-dimensional.

It is also interesting to consider *b*-analogs of classical Toeplitz operators.

Thanks for listening!

References:

- G. A. Mendoza: *Complex b-manifolds*, Contemp. Math. (2014).
- G. Scott: *The geometry of b^k manifolds*, J. Symplectic Geom. (2016).