Poisson reductions of master integrable systems on doubles of compact Lie groups

László Fehér, University of Szeged and Wigner RCP, Budapest

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Let us remember some celebrated integrable many-body Hamiltonians (of Calogero–Moser–Sutherland–Ruijsenaars–Schneider type):

$$H_{\mathrm{trig-Suth}}(q,p) \equiv \frac{1}{2}\sum_k p_k^2 + \frac{1}{2}\sum_{j\neq k}\frac{x^2}{\sin^2(q_j-q_k)}$$

$$H_{\mathrm{spin-Suth}}(q,p,\xi) = \frac{1}{2}\sum_{k=1}^n p_k^2 + \frac{1}{8}\sum_{j\neq k}\frac{|\xi_{jk}|^2}{\sin^2(q_j-q_k)},$$

where $\xi \in \mathfrak{u}(n)^*$, with zero diagonal part. This gives the standard (spinless) Sutherland model on a symplectic leaf, where $\xi_{jk} = \mathrm{i}x(\delta_{j,k}-1)$ with 'coupling constant' $x > 0$.

$$H_{\mathrm{trig-RS}} = \sum_{k=1}^n (\cosh p_k)\prod_{j\neq k}\left[1+\frac{\sinh^2 x}{\sin^2(q_j-q_k)}\right]^{\frac{1}{2}}$$

$$H_{\mathrm{compact-RS}} = \sum_{k=1}^n (\cos p_k)\sqrt{\prod_{j\neq k}\left[1-\frac{\sin^2 x}{\sin^2(q_j-q_k)}\right]}$$

All these models come by Hamiltonian reduction from a 'classical double' associated with SU(n).

Let G be a compact, connected (and simply connected) Lie group whose Lie algebra $\mathcal{G}$ is simple. After exploring a lot of specific reductions in the past, recently I investigated the most general reductions of three phase spaces built on G. The first is the cotangent bundle

$$\mathcal{M}_1 = T^*G \simeq G \times \mathcal{G}, \text{ with } \mathcal{G} := \mathrm{Lie}(G) \simeq \mathcal{G}^*,$$

i.e., the phase space of a point particle moving on G. The second is the 'Heisenberg double'

$$\mathcal{M}_2 = G \times \mathfrak{P}, \quad \mathfrak{P} := e^{\mathrm{i}\mathcal{G}} \subset G^{\mathbb{C}},$$

which is a symplectic manifold built on multiplicative Poisson structures on G and its Poisson–Lie dual. Example: $G = SU(n)$, $G^{\mathbb{C}} = SL(n, \mathbb{C})$, $\mathfrak{P} = \{X \in SL(n, \mathbb{C}) \mid X^\dagger = X,\ X \text{ has positive eigenvalues}\}$. The third unreduced phase space is the so-called quasi-Hamiltonian double

$$\mathcal{M}_3 = G \times G.$$

Moduli space of flat G-connections on punctured torus is $\mathcal{M}_3/G$.

The smooth real functions $C^\infty(\mathcal{M}_i)$ carry a Poisson bracket for $i = 1, 2$ and a quasi-Poisson bracket for $i = 3$. The group acts on all three phase spaces by 'diagonal conjugations', i.e., by the diffeomorphisms

$$\mathcal{A}^i_\eta : (x, y) \mapsto (\eta x \eta^{-1}, \eta y \eta^{-1}), \quad \forall x, y \in \mathcal{M}_i,\ \eta \in G.$$

In all three cases the G-invariant functions, $C^\infty(\mathcal{M}_i)^G$, form closed Poisson algebras. We identify $C^\infty(\mathcal{M}_i)^G$ with the smooth functions on the quotient

$$\mathcal{M}_i^{\text{red}} \equiv \mathcal{M}_i / G \quad \text{(in words: the set of } G\text{-orbits in } \mathcal{M}_i\text{).}$$

Thus the orbit space becomes a (singular) Poisson space. Our first goal is to derive a convenient description of the reduced Poisson algebras

$$(C^\infty(\mathcal{M}_i)^G, \{\ ,\ \}_{\mathcal{M}_i}).$$

That is, we wish to describe them in terms of variables as close to coordinates on $\mathcal{M}_i^{\text{red}}$ as possible.

Remark: The main property of the so called quasi-Poisson bracket is that the Jacobi identity is not always valid, but it holds for G-invariant functions.

We have a pair of degenerate integrable Hamiltonian systems on $\mathcal{M}_i$ (for each $i = 1, 2, 3$). To describe these, consider the rings of G-invariant functions:

$$C^\infty(\mathcal{G})^G, \qquad C^\infty(G)^G, \quad C^\infty(\mathfrak{P})^G.$$

Then let π_1^i and π_2^i denote the projections from $\mathcal{M}_i$ onto the first and second factors of this direct product space. By using the projections, we can pull-back the relevant two rings of invariants to $\mathcal{M}_i$, and thus obtain two Abelian Poisson algebras on $\mathcal{M}_i$. In each case, the 'pull-back invariants' form a ring of functional dimension $r := \mathrm{rank}(\mathcal{G})$, and the ring of their joint constants of motion has functional dimension $\dim(\mathcal{M}_i) - r$. (We have $\mathrm{rank}(su(n)) = (n-1)$.)

This means that the pull-back invariants define a degenerate integrable Hamiltonian (or quasi-Hamiltonian) system. Their generic Liouville 'tori' have dimension $r < \frac{1}{2}\dim(\mathcal{M}_i) = \dim(G)$. We can also write the phase space flows generated by the pull-back invariants explicitly, and they are complete.

Our second goal is to characterize the reductions of the degenerate integrable systems living on $\mathcal{M}_i$ for $i = 1, 2, 3$.

Recall of degenerate integrability on symplectic and Poisson manifolds

Definition 1. Suppose that $\mathcal{M}$ is a **symplectic** manifold of dimension $2m$ with associated Poisson bracket $\{-,-\}$ and two distinguished subrings $\mathfrak{H}$ and $\mathfrak{F}$ of $C^\infty(\mathcal{M})$ satisfying the following conditions:

1. The ring $\mathfrak{H}$ has functional dimension r and $\mathfrak{F}$ has functional dimension s such that $r+s=\dim(\mathcal{M})$ and $r<m$.
2. Both $\mathfrak{H}$ and $\mathfrak{F}$ form Poisson subalgebras of $C^\infty(\mathcal{M})$, satisfying $\mathfrak{H}\subset\mathfrak{F}$ and $\{\mathcal{F},\mathcal{H}\}=0$ for all $\mathcal{F}\in\mathfrak{F}$, $\mathcal{H}\in\mathfrak{H}$.
3. The Hamiltonian vector fields of the elements of $\mathfrak{H}$ are complete.

Then, $(\mathcal{M},\{-,-\},\mathfrak{H},\mathfrak{F})$ is called a **degenerate integrable system of rank** r. The rings $\mathfrak{H}$ and $\mathfrak{F}$ are referred to as the ring of Hamiltonians and constants of motion, respectively. (If $r=1$, then this is the same as 'maximal superintegrability' of a single Hamiltonian.)

Definition 2. Consider a **Poisson** manifold $(\mathcal{M},\{-,-\})$ whose Poisson tensor has maximal rank $2m\leq\dim(\mathcal{M})$ on a dense open subset. Then, $(\mathcal{M},\{-,-\},\mathfrak{H},\mathfrak{F})$ is called a degenerate integrable system of rank r if conditions (1), (2), (3) of Definition 1 hold, and the Hamiltonian vector fields of the elements of $\mathfrak{H}$ span an r-dimensional subspace of the tangent space over a dense open subset of $\mathcal{M}$.

Integrability is generically inherited under reduction
A mechanism behind reduced integrability

Consider two G-manifolds $\mathcal{M}$ and $\mathfrak{C}$ and the natural projections $p_1 : \mathcal{M} \to \mathcal{M}/G$ and $p_2 : \mathfrak{C} \to \mathfrak{C}/G$. Suppose that p_1 and p_2 are smooth submersions, and that $\Psi : \mathcal{M} \to \mathfrak{C}$ is also a smooth, G-**equivariant**, surjective submersion. This induces a surjective submersion

$$\Psi_{\text{red}} : \mathcal{M}/G \to \mathfrak{C}/G \qquad \text{for which} \qquad p_2 \circ \Psi = \Psi_{\text{red}} \circ p_1.$$

Let V be a vector field on $\mathcal{M}$ that is projectable onto a vector field, V_{red}, on $\mathcal{M}/G$. In this case, **if Ψ is constant along the integral curves of V, then Ψ_{red} is constant along the integral curves of V_{red}.** Consequently, $\Psi^*(C^\infty(\mathfrak{C}))$ gives constants of motion for V, and $\Psi^*_{\text{red}}(C^\infty(\mathfrak{C}/G))$ gives constants of motion for V_{red}.

It turns out that this mechanism is applicable to all of our cases, after restriction to a certain dense open subset.

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad\Psi\quad} & \mathfrak{C} \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ \mathcal{M}/G & \xrightarrow{\quad\Psi_{\text{red}}\quad} & \mathfrak{C}/G \end{array}$$

I extracted this method from Reshetikhin's paper: *Degenerate integrability of spin Calogero–Moser systems and the duality with the spin Ruijsenaars systems*. Lett. Math. Phys. **63** (2003) 55-71.

Plan of the rest of the talk

- The (well known) case of the cotangent bundle T^*G
- Just flash: Reduced equations of motion in the other cases
- Spin RS type models from Heisenberg doubles
- Time permitting: Reduction of the quasi-Poisson double
- Conclusion

The example of the cotangent bundle

The canonical Poisson bracket on the cotangent bundle

$$\mathcal{M} := \mathcal{M}_1 = G \times \mathcal{G} = \{(g, J) \mid g \in G,\ J \in \mathcal{G}\} \text{ has the form}$$

$$\{\mathcal{F}, \mathcal{H}\}(g, J) = \langle \nabla_1 \mathcal{F}, d_2 \mathcal{H} \rangle - \langle \nabla_1 \mathcal{H}, d_2 \mathcal{F} \rangle + \langle J, [d_2 \mathcal{F}, d_2 \mathcal{H}] \rangle,$$

where the $\mathcal{G}$-valued derivatives are taken at (g, J). Here, $\langle X, Y \rangle$ is the Cartan-Killing inner product on $\mathcal{G}$. The derivative $d_2 \mathcal{F} \in \mathcal{G}$ w.r.t. the second variable $J \in \mathcal{G}$ is the usual gradient, while the derivative $\nabla_1 \mathcal{F} \in \mathcal{G}$ w.r.t. first variable $g \in G$ is defined by

$$\left. \frac{d}{dt} \right|_{t=0} \mathcal{F}(e^{tX} g e^{tY}, J) =: \langle X, \nabla_1 \mathcal{F}(g, J) \rangle + \langle Y, \nabla_1' \mathcal{F}(g, J) \rangle, \forall X, Y \in \mathcal{G}.$$

The equations of motion generated by $\mathcal{H} = \pi_2^* \varphi$ with $\varphi \in C^\infty(\mathcal{G})^G$ read

$$\dot{g} = (d\varphi(J))g,\ \dot{J} = 0 \implies (g(t), J(t)) = (\exp(td\varphi(J(0)))g(0), J(0)).$$

The constants of motions are arbitrary functions of J and $g^{-1}Jg$. These constants of motion engender the G-equivariant map $\Psi : \mathcal{M} \ni (g, J) \mapsto (J, g^{-1}Jg) \in \mathcal{G} \times \mathcal{G}$,

$$\mathfrak{C} := \Psi(\mathcal{M}) = \{(J, \tilde{J}) \in \mathcal{G} \times \mathcal{G} \mid \chi(J) = \chi(\tilde{J}),\ \forall \chi \in C^\infty(\mathcal{G})^G\}.$$

This implies degenerate integrability on $\mathcal{M}$, and entails that the previously outlined mechanism behind reduced integrability is applicable after a suitable restriction.

Let $\mathcal{G}_0 < \mathcal{G}$ be a maximal Abelian subalgebra and $G_0 < G$ the corresponding Lie subgroup, i.e., a maximal torus. In the $SU(n)$ case we take the diagonal subalgebra and subgroup: $G_0 = \{\text{diag}(\tau_1, \ldots, \tau_n) \in SU(n) \mid |\tau_i| = 1\}$. Then G_0^{reg} and G^{reg} consist of unitary matrices with n distinct eigenvalues.

Reduced Poisson brackets and dynamics

We characterize the reduced system using a partial gauge fixing. Define

$$\mathcal{M}^{\mathrm{reg}} := \{(g,J)\in\mathcal{M} \mid g\in G^{\mathrm{reg}}\}, \quad \mathcal{M}_0^{\mathrm{reg}} := \{(Q,J)\in\mathcal{M} \mid Q\in G_0^{\mathrm{reg}}\}.$$

Then $\mathcal{M}^{\mathrm{reg}}/G \equiv \mathcal{M}_0^{\mathrm{reg}}/\mathfrak{N}$ and restriction of functions yields the isomorphism

$$C^\infty(\mathcal{M}^{\mathrm{reg}})^G \Longleftrightarrow C^\infty(\mathcal{M}_0^{\mathrm{reg}})^{\mathfrak{N}},$$

where $\mathfrak{N} < G$ is the normalizer of G_0 in G ($\mathfrak{N}/G_0$ is the Weyl group). That is, $\mathfrak{N}$ is the 'group of residual gauge transformations'.

By transferring the PB from $C^\infty(\mathcal{M}^{\mathrm{reg}})^G$ to $C^\infty(\mathcal{M}_0^{\mathrm{reg}})^{\mathfrak{N}}$, we get

$$\{F,H\}_{\mathrm{red}}(Q,J) = \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle + \langle J, [d_2 F, d_2 H]_{\mathcal{R}(Q)}\rangle,$$

with $[X,Y]_{\mathcal{R}} \equiv [\mathcal{R}X, Y] + [X, \mathcal{R}Y]$. The 'reduced evolution equations' can be written on $\mathcal{M}_0^{\mathrm{reg}}$ as

$$\dot{Q} = (d\varphi(J))_0 Q, \qquad \dot{J} = [\mathcal{R}(Q) d\varphi(J), J].$$

The subscript zero refers to the orthogonal decomposition $\mathcal{G} = \mathcal{G}_0 + \mathcal{G}_\perp$, and $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G})$ is the basic trigonometric solution of the modified classical dynamical Yang–Baxter equation. $\mathcal{R}(Q)$ vanishes on $\mathcal{G}_0$ and, writing $Q = \exp(\mathrm{i}q)$ with $q \in \mathrm{i}\mathcal{G}_0^{\mathrm{reg}}$, is given on $\mathcal{G}_\perp$ by $\mathcal{R}(Q) = \frac{1}{2}\coth(\frac{\mathrm{i}}{2}\mathrm{ad}_q)$. For $SU(n)$, $(\mathcal{R}(Q)X)_{jk} = \frac{1}{2}(1-\delta_{jk})X_{jk}\coth(\frac{\mathrm{i}}{2}(q_j - q_k))$. These evolution equations are unique up to residual gauge transformations.

The (well known) spin Sutherland interpretation

Parametrize $J \in \mathcal{G}$ according to

$$J = -\mathrm{i}p + \sum_{\alpha>0}\left(\frac{\xi_\alpha}{e^{-\mathrm{i}\alpha(q)}-1}E_\alpha - \frac{\xi_\alpha^*}{e^{\mathrm{i}\alpha(q)}-1}E_{-\alpha}\right), \ p \in \mathrm{i}\mathcal{G}_0,$$

and take $\varphi(J) = -\frac{1}{2}\langle J, J\rangle$. Then we get

$$H_{\mathrm{spin-Suth}}(q,p,\xi) = -\frac{1}{2}\langle J, J\rangle = \frac{1}{2}\langle p, p\rangle + \frac{1}{4}\sum_{\alpha>0}\frac{|\xi_\alpha|^2}{\sin^2(\alpha(q)/2)},$$

which is a standard spin Sutherland Hamiltonian. Here, we use the Killing form and the root space decomposition of the complexified Lie algebra $\mathcal{G}^{\mathbb{C}}$, with the set of positive roots $\{\alpha\}$ and corresponding root vectors E_α. The 'collective spin variable' $\xi = \sum_{\alpha>0}(\xi_\alpha E_\alpha - \xi_\alpha^* E_{-\alpha}) \in \mathcal{G}_\perp$ matters up to conjugations by the maximal torus G_0, i.e., ξ belongs to the reduction of $\mathcal{G}^* \simeq \mathcal{G}$ with respect to the Hamiltonian action of G_0 on $\mathcal{G}^*$, at zero moment map value. There is also a residual Weyl group symmetry.

In the $SU(n)$ case, $H_{\mathrm{spin-Suth}}(q,p,\xi) = \frac{1}{2}\sum_{k=1}^n p_k^2 + \frac{1}{8}\sum_{j\neq k}\frac{|\xi_{jk}|^2}{\sin^2((q_j-q_k)/2)}$, and one gets the standard (spinless) Sutherland model on a symplectic leaf, where $\xi_{jk} = \mathrm{i}x(\delta_{j,k}-1)$ with 'coupling constant' $x>0$. It describes an integrable system of n 'point particles' moving on the unit circle.

The other degenerate integrable system on $\mathcal{M} = T^*G$ is generated by the invariants

$$\mathcal{H} = \pi_1^* h, \quad (\mathcal{H}(g,J) = h(g)) \quad \text{for all} \quad h \in C^\infty(G)^G.$$

They span an Abelian Poisson algebra of functional dimension $r = \text{rank}(\mathcal{G})$. The unreduced evolution equations and their flows read

$$\dot{g} = 0, \ \dot{J} = -\nabla h(g), \quad (g(t), J(t)) = (g(0), J(0) - t\nabla h(g(0)).$$

The constants of motion are arbitrary functions of the pair (g, Φ), with the moment map $\Phi(g,J) = J - \tilde{J}$. To characterize the reduced system, we now consider an other dense open subset and alternative gauge slice

$$\tilde{\mathcal{M}}^{\text{reg}} := \{(g,J) \in \mathcal{M} \mid J \in \mathcal{G}^{\text{reg}}\}, \qquad \tilde{\mathcal{M}}_0^{\text{reg}} := \{(g,\lambda) \in \mathcal{M} \mid \lambda \in \mathcal{G}_0^{\text{reg}}\}.$$

We find the reduced Poisson bracket

$$\begin{aligned} \{F,H\}\tilde{}_{\text{red}}(g,\lambda) =& \langle \nabla_1 F, d_2 H\rangle - \langle \nabla_1 H, d_2 F\rangle \\ &+ \langle \nabla_1' F, r(\lambda)\nabla_1' H\rangle - \langle \nabla_1 F, r(\lambda)\nabla_1 H\rangle \end{aligned}$$

(containing a rational dynamical r-matrix) and the reduced evolution equations

$$\dot{\lambda} = -(\nabla h(g))_0, \qquad \dot{g} = [g, r(\lambda)\nabla h(g)].$$

Concretely, $r(\lambda) \in \mathrm{End}(\mathcal{G})$ is the standard rational dynamical r-matrix:

$$r(\lambda)X = (\mathrm{ad}_\lambda)^{-1}(X_\perp), \quad \forall X = (X_0 + X_\perp) \in (\mathcal{G}_0 + \mathcal{G}_\perp).$$

For $\mathcal{G} = su(n)$, $\lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, and $(r(\lambda)X)_{jk}) = X_{jk}/(\lambda_j - \lambda_k)$.

For $G = \mathrm{SU}(n)$, on a special symplectic leaf, the reduced system gives the so called Ruijsenaars dual of the trigonometric Sutherland model, with main Hamiltonian

$$\widetilde{H}_{\mathrm{rat-RS}}(\lambda, \varphi) = \sum_{k=1}^{n} (\cos \varphi_k) \prod_{j \neq k} \left[1 - \frac{x^2}{(\lambda_k - \lambda_j)^2} \right]^{\frac{1}{2}}.$$

The two kinds of models exhibit 'position–action' duality: $C^\infty(\mathcal{G})^G$ and $C^\infty(G)^G$ reduce to functions of action variables and functions of position variables in the two kinds of models, respectively, but their role is interchanged.

The form of the reduced equations of motion in the other cases

First, take $\mathcal{H} = \pi_2^* \phi \in C^\infty(\mathcal{M}_2)$, $\phi \in C^\infty(\mathfrak{P})^G$. Then the evolution of the reduced variables $(Q, L) \in G_0^{\text{reg}} \times \mathfrak{P}$ is governed by

$$\dot{Q} = (\mathcal{D}\phi(L))_0 Q, \qquad \dot{L} = [\mathcal{R}(Q)\mathcal{D}\phi(L), L],$$

with derivative $\mathcal{D}\phi(L) \in \mathcal{G}$.

Second, for $\mathcal{H} = \pi_1^* h \in C^\infty(\mathcal{M}_2)$, $h \in C^\infty(G)^G$, the reduced variables $(g, P) \in G \times \exp(\mathrm{i}\mathcal{G}_0^{\text{reg}})$ satisfy

$$\dot{P} = -2\mathrm{i}(\nabla h(g))_0 P, \qquad \dot{g} = 2\mathrm{i}[g, \mathcal{R}(P)\nabla h(g)].$$

Finally, for $\mathcal{H} = \pi_2^* h \in C^\infty(\mathcal{M}_3)$ with $h \in C^\infty(G)^G$, the reduced variables $(Q, g) \in G_0^{\text{reg}} \times G$ obey

$$\dot{Q} = -(\nabla h(g))_0 Q, \qquad \dot{g} = [g, \mathcal{R}(Q)\nabla h(g)].$$

We also have the reduced Poisson brackets in all cases, and can prove degenerate integrability after restriction to a suitable dense open subset of the reduced phase space.

In the $SU(n)$ case, restriction to a special symplectic leaf in $\mathcal{M}_2/G$ reproduces the trigonometric Ruijsenaars–Schneider model and its action-angle dual. The main Hamiltonians of these specially restricted reduced systems are

$$H_{\text{trigo-RS}} = \sum_{k=1}^{n} (\cosh p_k) \prod_{j \neq k} \left[1 + \frac{\sinh^2 x}{\sin^2(q_k - q_j)} \right]^{\frac{1}{2}},$$

which represent a deformation of the Sutherland Hamiltonian, and its dual pair

$$\hat{H}_{\text{trigo-RS}} = \sum_{k=1}^{n} (\cos \hat{q}_k) \prod_{j \neq k} \left[1 - \frac{\sinh^2 x}{\sinh^2(\hat{p}_k - \hat{p}_j)} \right]^{\frac{1}{2}}.$$

Still for $G = \mathrm{SU}(n)$, a special symplectic leaf in $\mathcal{M}_3/G$ carries the compactified trigonometric Ruijsenaars–Schneider model, whose main Hamiltonian reads (on a dense subset)

$$H_{\text{compact-RS}} = \sum_{k=1}^{n} (\cos p_k) \sqrt{\prod_{j \neq k} \left[1 - \frac{\sin^2 x}{\sin^2(q_k - q_j)} \right]}.$$

Spin RS type models from the Heisenberg double $\mathcal{M}_2 = G \times \mathfrak{P} =: \mathbb{M}$

Let us realize $\mathcal{G}$ as a compact real form of a complex simple Lie algebra $\mathcal{G}^{\mathbb{C}}$. Using positive roots associated with the Cartan subalgebra $\mathcal{G}_0^{\mathbb{C}}$, consider the triangular decompositions $\mathcal{G}^{\mathbb{C}} = \mathcal{G}_<^{\mathbb{C}} + \mathcal{G}_0^{\mathbb{C}} + \mathcal{G}_>^{\mathbb{C}}$. We also consider a corresponding connected and simply connected Lie group $G^{\mathbb{C}}$. The realification $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ of $\mathcal{G}^{\mathbb{C}}$ decomposes as the vector space direct sum

$$\mathcal{G}_{\mathbb{R}}^{\mathbb{C}} = \mathcal{G} + \mathcal{B} \quad \text{with} \quad \mathcal{B} := \mathrm{i}\mathcal{G}_0 + \mathcal{G}_>^{\mathbb{C}}.$$

$\mathcal{G}$ and $\mathcal{B}$ are isotropic subalgebras with respect to the invariant, symmetric, non-degenerate, real bilinear form $\langle -, - \rangle_{\mathbb{I}}$ on $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ defined by the imaginary part of the complex Killing form of $\mathcal{G}^{\mathbb{C}}$. If $\mathcal{G} = su(n)$, then $X \in \mathcal{B}$ is upper-triangular with real diagonal entries. For any $Z = Z_1 + \mathrm{i}Z_2$ in $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$, with $Z_1, Z_2 \in \mathcal{G}$, we let $Z^\dagger := -Z_1 + \mathrm{i}Z_2$.

For a real $f \in C^\infty(G)$ we define its $\mathcal{B}$-valued left- and right-derivatives by

$$\langle \mathcal{D}f(g), X \rangle_{\mathbb{I}} + \langle \mathcal{D}' f(g), Y \rangle_{\mathbb{I}} := \frac{d}{dt}\Big|_{t=0} f(e^{tX} g e^{tY}), \quad \forall X, Y \in \mathcal{G}.$$

For a real function $\phi \in C^\infty(\mathfrak{P})$ we define its $\mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$-valued derivative $\mathcal{D}\phi$ by

$$\langle X, \mathcal{D}\phi(L) \rangle_I := \frac{d}{dt}\Big|_{t=0} \phi(e^{tX} L e^{tX^\dagger}) \text{ and } \langle Y, \mathcal{D}\phi(L) \rangle_I := \frac{d}{dt}\Big|_{t=0} \phi(e^{tY} L e^{-tY})$$

$\forall X \in \mathcal{B}$ and $Y \in \mathcal{G}$. The first relation determines $(\mathcal{D}\phi(L))_{\mathcal{G}}$ and the second one $(\mathcal{D}\phi(L))_{\mathcal{B}}$. Incidentally, all information about $\mathcal{D}\phi$ is contained in the $\mathcal{G}$-component.

The phase space $\mathbb{M} = G \times \mathfrak{P}$ carries the following (symplectic) Poisson structure:

$$\{\mathcal{F}, \mathcal{H}\}(g, L) = \langle \mathcal{D}_2\mathcal{F}, (\mathcal{D}_2\mathcal{H})_{\mathcal{G}}\rangle_{\mathbb{I}} - \left\langle g\mathcal{D}_1'\mathcal{F}g^{-1}, \mathcal{D}_1\mathcal{H}\right\rangle_{\mathbb{I}} + \langle \mathcal{D}_1\mathcal{F}, \mathcal{D}_2\mathcal{H}\rangle_{\mathbb{I}} - \langle \mathcal{D}_1\mathcal{H}, \mathcal{D}_2\mathcal{F}\rangle_{\mathbb{I}},$$

where the derivatives of $\mathcal{F}, \mathcal{H} \in C^\infty(\mathbb{M})$ are evaluated at $(g, L) \in \mathbb{M}$. The Hamiltonian $\mathcal{H} = \pi_2^*\phi$, with $\phi \in C^\infty(\mathfrak{P})^G$, generates the evolution equation

$$\dot{g} = (\mathcal{D}\phi(L))g,\ \dot{L} = 0,\ \text{solved by } (g(t), L(t)) = (\exp\left(t\mathcal{D}\phi(L(0))\right) g(0), L(0)).$$

Therefore $\pi_2^* C^\infty(\mathfrak{P})^G$ forms an Abelian Poisson algebra of rank r, and the map

$$\Psi : \mathbb{M} \ni (g, L) \mapsto (L, g^{-1}Lg) \in \mathfrak{P} \times \mathfrak{P}$$

is constant along its flows. The image of Ψ is the subset $\mathfrak{C}$ of $\mathfrak{P} \times \mathfrak{P}$ defined by

$$\mathfrak{C} := \{(L, \tilde{L}) \in \mathfrak{P} \times \mathfrak{P} \mid \chi(L) = \chi(\tilde{L}),\ \forall \chi \in C^\infty(\mathfrak{P})^G\}.$$

$\mathfrak{H} := \pi_2^*\left(C^\infty(\mathfrak{P})^G\right)$ and $\mathfrak{F} := \Psi^*\left(C^\infty(\mathfrak{P} \times \mathfrak{P})\right)$ yield a degenerate integrable system.

Considering $\mathbb{M}^{\mathrm{reg}} := \{(g, L) \in \mathbb{M} \mid Q \in G^{\mathrm{reg}}\}$ and $\mathbb{M}_0^{\mathrm{reg}} := \{(Q, L) \in \mathbb{M} \mid Q \in G_0^{\mathrm{reg}}\}$, we have the isomorphism $C^\infty(\mathbb{M}^{\mathrm{reg}})^G \Longleftrightarrow C^\infty(\mathbb{M}_0^{\mathrm{reg}})^{\mathfrak{N}}$. This leads to the reduced Poisson bracket on $C^\infty(\mathbb{M}_0^{\mathrm{reg}})^{\mathfrak{N}}$:

$$\{F, H\}_{\mathrm{red}}(Q, L) = \langle \mathcal{D}_1 F, \mathcal{D}_2 H\rangle_{\mathbb{I}} - \langle \mathcal{D}_1 H, \mathcal{D}_2 F\rangle_{\mathbb{I}} + \langle \mathcal{R}(Q)(\mathcal{D}_2 H)_{\mathcal{G}}, \mathcal{D}_2 F\rangle_{\mathbb{I}} - \langle \mathcal{R}(Q)(\mathcal{D}_2 F)_{\mathcal{G}}, \mathcal{D}_2 F\rangle_{\mathbb{I}}$$

The derivatives $\mathcal{D}_1 F \in \mathcal{B}_0$ and $\mathcal{D}_2 F \in \mathcal{G}_{\mathbb{R}}^{\mathbb{C}}$ are taken at (Q, L), and $\mathcal{R}(Q) \in \mathrm{End}(\mathcal{G})$ is the standard dynamical r-matrix.

One recovers the reduced evolution equations by using $H = \pi_2^*\phi$, with $\phi \in C^\infty(\mathfrak{P})^G$.

For the system on $\mathbb{M}$ based on $\pi_1^*\left(C^\infty(G)^G\right)$ and its reduction, as well as for the reductions of the quasi-Hamiltonian double, see my paper arXiv:2208.03728.

Canonically conjugate pairs and 'spin' variables. Let B_0 and B_+ be the subgroups of B associated with the subalgebras in $\mathcal{B} = \mathcal{B}_0 + \mathcal{B}_>$. Any $b \in B$ is uniquely decomposed as $b = e^p b_+$ with $p \in \mathcal{B}_0$, $b_+ \in B_+$ and any $L \in \mathfrak{P}$ can be written as $L = bb^\dagger$ (where $X^\dagger = -\theta(X)$ for $X \in \mathcal{G}_\mathbb{R}^\mathbb{C}$ with the Cartan involution θ). Then, we introduce new variables by means of the map

$$\zeta : \mathbb{M}_0^{\mathrm{reg}} \to G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+$$

$$\zeta : (Q, L = e^p b_+ b_+^\dagger e^p) \mapsto (Q, p, \lambda) \quad \text{with} \quad \lambda := b_+^{-1} Q^{-1} b_+ Q.$$

The map ζ is a diffeomorphism.

In terms of the new variables introduced via the map ζ, the reduced Poisson bracket acquires the following 'decoupled form':

$$\{F, H\}^{\mathrm{red}}(Q, p, \lambda) = \langle D_Q F, d_p H \rangle_\mathbb{I} - \langle D_Q H, d_p F \rangle_\mathbb{I} + \langle \lambda D'_\lambda F \lambda^{-1}, D_\lambda H \rangle_\mathbb{I},$$

where the derivatives of $F, H \in C^\infty(G_0^{\mathrm{reg}} \times \mathcal{B}_0 \times B_+)^{\mathfrak{N}}$ are taken at (Q, p, λ).

Using the identification $(\mathcal{B}_+)^* \simeq \mathcal{G}_\perp$, the derivatives $D_\lambda F, D'_\lambda F \in \mathcal{G}_\perp$ are defined by

$$\langle X_+, D_\lambda F(Q, p, \lambda) \rangle_\mathbb{I} + \langle X'_+, D'_\lambda F(Q, p, \lambda) \rangle_\mathbb{I} = \frac{d}{dt}\bigg|_{t=0} F(Q, p, e^{tX_+} \lambda e^{tX'_+}), \;\; \forall X_+, X'_+ \in \mathcal{B}_+.$$

The last term represents the reduction of the Poisson–Lie group $B = G^*$ with respect to the maximal torus G_0, at the zero value of the moment map for the G_0-action on $(B, \{-,-\}_B)$. **This is very similar to the variables underlying the spin Sutherland models coming from T^*G.**

Interpretation as spin RS model: Consider the new variable $\lambda = b_+^{-1} Q^{-1} b_+ Q$ using

$$\lambda = e^{\sigma}, \quad b_+ = e^{\beta}, \quad \sigma = \sum_{\alpha>0} \sigma_\alpha E_\alpha, \quad \beta = \sum_{\alpha>0} \beta_\alpha E_\alpha, \quad Q = e^{\mathrm{i}q}.$$

We find β_α in terms of σ and $e^{\mathrm{i}q}$: $\beta_\alpha = \frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)}-1} + \sum_{k\geq 2} \sum_{\varphi_1,\dots,\varphi_k} f_{\varphi_1,\dots,\varphi_k}(e^{\mathrm{i}q}) \sigma_{\varphi_1} \dots \sigma_{\varphi_k}$, where $\alpha = \varphi_1 + \dots + \varphi_k$ and $f_{\varphi_1,\dots,\varphi_k}$ depends rationally on $e^{\mathrm{i}q}$.

Take any finite dimensional irreducible representation $\rho : G^{\mathbb{C}} \to \mathrm{SL}(V)$. Introduce an inner product on V so that the dagger, $K^\dagger = \Theta(K^{-1})$, becomes the usual adjoint. Then, the (normalized) character $\phi^\rho(L) = \mathrm{tr}_\rho(L) := c_\rho \mathrm{tr}\rho(L)$ gives an element of $C^\infty(\mathfrak{P})^G$. (Here, c_ρ is a constant, so that $\mathrm{tr}_\rho(XY) := c_\rho \mathrm{tr}(\rho(X)\rho(Y)) = \langle X, Y \rangle$, $\forall X, Y \in \mathcal{G}^{\mathbb{C}}$.)

Using the 'decoupled variables' (Q, p, σ), $H^\rho := \mathrm{tr}_\rho(e^p b_+ b_+^\dagger e^p)$ can be expanded as

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = \mathrm{tr}_\rho \left(e^{2p} \left(\mathbf{1}_\rho + \frac{1}{4} \sum_{\alpha>0} \frac{|\sigma_\alpha|^2 E_\alpha E_{-\alpha}}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*) \right) \right).$$

By expanding e^{2p},

$$H^\rho(e^{\mathrm{i}q}, p, \sigma) = c_\rho \mathrm{dim}_\rho + 2\mathrm{tr}_\rho(p^2) + \frac{1}{2} \sum_{\alpha>0} \frac{1}{|\alpha|^2} \frac{|\sigma_\alpha|^2}{\sin^2(\alpha(q)/2)} + \mathrm{o}_2(\sigma, \sigma^*, p).$$

The leading term matches the spin Sutherland Hamiltonian $H_{\mathrm{spin-Suth}}$. The reduced Poisson brackets and the Lax matrix are also deformations of those appearing in the spin Sutherland models. Indeed, in any representation, we have

$$L(e^{\mathrm{i}q}, p, \sigma) = 1 + 2p + \sum_{\alpha>0} \left(\frac{\sigma_\alpha}{e^{-\mathrm{i}\alpha(q)} - 1} E_\alpha + \frac{\sigma_\alpha^*}{e^{\mathrm{i}\alpha(q)} - 1} E_{-\alpha} \right) + \mathrm{o}(\sigma, \sigma^*, p),$$

which matches the standard, $\mathrm{i}\mathcal{G}$-valued, spin Sutherland Lax matrix.

Explicit formulas for $G^{\mathbb{C}} = SL(n,\mathbb{C})$: Now parametrize $b_+ \in B$ by its matrix elements. We have $b = e^p b_+$, and can find b_+ from the relation

$$Q^{-1} b_+ Q = b_+ \lambda,$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_n) \in G_0^{\mathrm{reg}}$, $\lambda \in B_+$ is the constrained 'spin' variable and b_+ is an upper triangular matrix with unit diagonal.

Introducing $\mathcal{I}_{a,a+j} := \frac{1}{Q_{a+j}Q_a^{-1} - 1}$, we have $(b_+)_{a,a+1} = \mathcal{I}_{a,a+1}\lambda_{a,a+1}$, and, for $k = 2, \ldots, n-a$, the matrix element $(b_+)_{a,a+k}$ equals

$$\mathcal{I}_{a,a+k}\lambda_{a,a+k} + \sum_{\substack{m=2,\ldots,k \\ (i_1,\ldots,i_m)\in\mathbb{N}^m \\ i_1+\cdots+i_m=k}} \prod_{\alpha=1}^{m} \mathcal{I}_{a,a+i_1+\cdots+i_\alpha} \lambda_{a+i_1+\cdots+i_{\alpha-1},a+i_1+\cdots+i_\alpha}.$$

Then $H = \mathrm{tr}(bb^\dagger)$ gives

$$H(e^{\mathrm{i}q}, p, \lambda) = \sum_{a=1}^{n} e^{2p_a} + \frac{1}{4}\sum_{a=1}^{n-1} e^{2p_a} \sum_{k=1}^{n-a} \frac{|\lambda_{a,a+k}|^2}{\sin^2((q_{a+k}-q_a)/2)} + \mathrm{o}_2(\lambda, \lambda^\dagger).$$

Restricting λ to a minimal dressing orbit of $SU(n)$ results in the standard (spinless) real, trigonometric Ruijsenaars–Schneider model.

On the reduction of the quasi-Hamiltonian double $\mathcal{M}_3 = G \times G =: \mathfrak{D}$

The starting point is the quasi-Poisson bracket

$$\begin{aligned} 2\{\mathcal{F},\mathcal{H}\} = &\langle \nabla_1'\mathcal{H}, \nabla_2\mathcal{F}\rangle - \langle \nabla_2\mathcal{H}, \nabla_1'\mathcal{F}\rangle + \langle \nabla_1\mathcal{H}, \nabla_2'\mathcal{F}\rangle - \langle \nabla_2'\mathcal{H}, \nabla_1\mathcal{F}\rangle \\ &+ \langle \nabla_2\mathcal{H}, \nabla_1\mathcal{F}\rangle - \langle \nabla_1\mathcal{H}, \nabla_2\mathcal{F}\rangle + \langle \nabla_1'\mathcal{H}, \nabla_2'\mathcal{F}\rangle - \langle \nabla_2'\mathcal{H}, \nabla_1'\mathcal{F}\rangle \\ &+ \langle \nabla_1\mathcal{H}, \nabla_1'\mathcal{F}\rangle - \langle \nabla_1'\mathcal{H}, \nabla_1\mathcal{F}\rangle + \langle \nabla_2'\mathcal{H}, \nabla_2\mathcal{F}\rangle - \langle \nabla_2\mathcal{H}, \nabla_2'\mathcal{F}\rangle. \end{aligned}$$

Consider arbitrary functions $\mathcal{F} \in C^\infty(\mathfrak{D})$ and $h \in C^\infty(G)^G$. Then we get

$$\{\mathcal{F}, \pi_2^* h\}(g_1, g_2) = -\langle \nabla_1'\mathcal{F}(g_1, g_2), \nabla h(g_2)\rangle.$$

Thus $\pi_2^* h$ induces the evolution equation $\dot{g}_1 = -g_1 \nabla h(g_2)$, $\dot{g}_2 = 0$, having the solution

$$(g_1(t), g_2(t)) = (g_1(0)\exp(-t\nabla h(g_2(0))), g_2(0)).$$

Therefore the ring $\pi_2^* C^\infty(G)^G$ forms an Abelian Poisson algebra, and g_2 as well as $g_1 g_2 g_1^{-1}$ are constants along all of the corresponding integral curves. Degenerate integrability and reduced degenerate integrability can be shown similarly to the cotangent bundle case, now using the equivariant map $\Psi : \mathfrak{D} \ni (g_1, g_2) \mapsto (g_1 g_2 g_1^{-1}, g_2) \in G \times G$,

$$\mathfrak{C} := \Psi(\mathfrak{D}) = \{(\tilde{g}, g) \in G \times G \mid \chi(g) = \chi(\tilde{g}),\ \forall \chi \in C^\infty(G)^G\}.$$

For $\mathcal{F}, \mathcal{H} \in C^\infty(\mathfrak{D})^G$, the quasi-Poisson bracket yields the **Poisson algebra**

$$2\{\mathcal{F},\mathcal{H}\} = \langle \nabla_1\mathcal{H}, \nabla_2\mathcal{F} + \nabla_2'\mathcal{F}\rangle - \langle \nabla_1\mathcal{F}, \nabla_2\mathcal{H} + \nabla_2'\mathcal{H}\rangle + \langle \nabla_2\mathcal{H}, \nabla_2'\mathcal{F}\rangle - \langle \nabla_2'\mathcal{H}, \nabla_2\mathcal{F}\rangle,$$

by virtue of the invariance property $\nabla_1\mathcal{F} - \nabla_1'\mathcal{F} + \nabla_2\mathcal{F} - \nabla_2'\mathcal{F} = 0$.

Introduce the submanifolds

$$\mathfrak{D}^{\text{reg}} := \{(g_1, g_2) \in \mathfrak{D} \mid g_1 \in G^{\text{reg}}\}, \qquad \mathfrak{D}_0^{\text{reg}} := \{(Q, g) \in \mathfrak{D} \mid Q \in G_0^{\text{reg}}\}.$$

Restriction of functions gives rise to the isomorphism $C^\infty(\mathfrak{D}^{\text{reg}})^G \Longleftrightarrow C^\infty(\mathfrak{D}_0^{\text{reg}})^{\mathfrak{N}}$. Any $F \in C^\infty(\mathfrak{D}_0^{\text{reg}})$ has the $\mathcal{G}_0$-valued derivative $\nabla_1 F$ and $\mathcal{G}$-valued derivatives $\nabla_2 F$, $\nabla_2' F$.

If $F, H \in C^\infty(\mathfrak{D}_0^{\text{reg}})^{\mathfrak{N}}$ are the restrictions of $\mathcal{F}, \mathcal{H} \in C^\infty(\mathfrak{D}^{\text{reg}})^G$, then the definition

$$\{F, H\}_{\text{red}}(Q, g) := \{\mathcal{F}, \mathcal{H}\}(Q, g), \quad \forall (Q, g) \in \mathfrak{D}_0^{\text{reg}},$$

leads to the formula

$$\{F, H\}_{\text{red}}(Q, g) = \langle \nabla_1 H, \nabla_2 F \rangle - \langle \nabla_1 F, \nabla_2 H \rangle + \langle \nabla_2' F, \mathcal{R}(Q) \nabla_2' H \rangle - \langle \nabla_2 F, \mathcal{R}(Q) \nabla_2 H \rangle.$$

The Hamiltonian $H(Q, g) = h(g)$, with $h \in C^\infty(G)^G$, generates the reduced equations of motion.

Conclusion and open questions

1. All our reduced systems are integrable in the degenerate sense, at least on a dense open subset of the smooth component, $\mathcal{M}_*/G$, of the full reduced phase space $\mathcal{M}/G$. My proof of reduced integrability relies on the 'restricted diagram' shown below, where $\mathfrak{C}_* \subset \mathfrak{C}_{\rm reg}$ contains the principal orbits of G in $\mathfrak{C}_{\rm reg} \subset \mathfrak{C}$, and $\mathcal{M}_{**} \subset \mathcal{M}_* \subset \mathcal{M}$ is its preimage. $\mathfrak{C}_{\rm reg}$ is a dense open subset of $\mathfrak{C} := \Psi(\mathcal{M})$ and is a smooth manifold. All the 4 maps in the diagram are smooth submersions and (quasi)Poisson maps.

$$
\begin{array}{ccc}
\mathcal{M}_{**} & \xrightarrow{\quad \Psi \quad} & \mathfrak{C}_* \\
\Big\downarrow p_1 & & \Big\downarrow p_2 \\
\mathcal{M}_{**}/G & \xrightarrow{\quad \Psi_{\rm red} \quad} & \mathfrak{C}_*/G
\end{array}
$$

This is a variant (refinement) of the method used by Reshetikhin in several papers.

2. How to modify the proof to extend the claims from $\mathcal{M}_{**}/G$ to $\mathcal{M}_*/G$?

3. What about integrability on arbitrary symplectic leaves of $\mathcal{M}/G$?

4. Quantization of the novel spin RS type models by quantum Hamiltonian reduction?