Kostant-Souriau-Odzijewicz quantization of a mechanical system whose classical phase space is a complex manifold[1]

Sorin Dragomir[2]

Abstract. The originally announced talk's purpose was to report on the calculation of the transition probability amplitudes $a_{0\overline{0}}(\zeta, z)$ from a state $z \in \Omega_n$ to a state $\zeta \in \Omega_n$ [for a mechanical system whose classical phase space is the Siegel domain $\Omega_n = \{\zeta \in \mathbb{C}^n \, : \, \mathrm{Im}(\zeta_n) > |\zeta'|^2\}$] once the classical states $z, \zeta \in \Omega_n$ are identified with the coherent states $\mathscr{K}(z), \mathscr{K}(\zeta) \in \mathbb{CP}(\mathscr{H})$, in the presence of the coherent state map $\mathscr{K} : \Omega_n \to \mathbb{CP}(\mathscr{H})$ built by A. Odzijewicz[3] essentially in terms of the weighted Bergman kernel of Ω_n corresponding to an admissible weight function $\gamma \in AW(\Omega)$ where $\mathscr{H} = L^2 H(\Omega_n , \gamma)$. The first two days (July 2nd-3rd, 2023) of the workshop revealed an audience spread through several areas of physics and mathematical physics, partially unaware of the complex analysis specific to the theory of reproducing kernel Hilbert spaces and their many applications, among older or newer. Consequently this speaker chose to integrate the slides of his talk with a blackboard exposition of some basic material (Bergman kernels, Fefferman's asymptotic expansion formula, etc.) on which the announced maters in the talk rely and which are strongly motivational for the talk. Only a few more advanced results found an actual blackboard space, while the more elementary notions and constructs were added to the present text after the lecture, in an effort to reach a more readable form of the text that would be made public online.

1. The Bergman kernel

Let $\Omega \subset \mathbb{C}^n$ be a domain, and let $f \in L^2 H(\Omega)$ be a holomorphic L^2 function $f : \Omega \to \mathbb{C}$. Let $A \subset \Omega$ be a compact subset, and let $\zeta \in A$. Let $\epsilon = (\epsilon_1 , \cdots , \epsilon_n)$ be a poliradius such that the closed polidisc $\overline{P}(\zeta, \epsilon) \subset \Omega$. As f is holomorphic, one may certainly expand

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f in convergent power series

$$f(z)=\sum_{|\alpha|=0}^{\infty} a_\alpha\,(z-\zeta)^\alpha$$

on $P(\zeta,\,\epsilon)$. If

$$\langle f,\,g\rangle_{L^2}=\int_\Omega f(z)\,\overline{g(z)}\;d\,\mu(z),\quad \|f\|=\langle f,\,f\rangle_{L^2}^{1/2}\,,$$

are respectively the L^2 inner product and L^2 norm on the Lebesgue space $L^2(\Omega)$ we conduct the estimates

$$\|f\|^2=\int_\Omega \big|f(z)\big|^2\;d\,\mu(z)\geq \int_{P(\zeta,\,\epsilon)}\big|f(z)\big|^2\;d\,\mu(z)=$$

$$=\int_{P(\zeta,\,\epsilon)}\sum_{\alpha,\,\beta} a_\alpha\,\overline{a}_\beta\,(z-\zeta)^\alpha\,(\overline{z}-\overline{\zeta})^\beta\;d\,\mu(z)=$$

[we may integrate term-by-term, as the convergence of the relevant series is uniform on $P(\zeta,\,\epsilon)$]

$$=\sum_{\alpha,\,\beta} a_\alpha\,\overline{a}_\beta\int_{P(\zeta,\,\epsilon)}(z-\zeta)^\alpha\,(\overline{z}-\overline{\zeta})^\beta\;d\,\mu(z)=$$

[by taking into account that the monomials

$$(z-\zeta)^\alpha,\quad (z-\zeta)^\beta,\quad \alpha\neq\beta\,,$$

are orthogonal]

$$=\sum_{\alpha}|a_\alpha|^2\int_{P(\zeta,\,\epsilon)}|z-\zeta|^{2\alpha}\;d\,\mu(z)\geq \big|a_{(0,\,\cdots,\,0)}\big|^2\,\mu\big(P(\zeta,\,\epsilon)\big)$$

or

$$\big|f(\zeta)\big|\leq \mu\big(P(\zeta,\,\epsilon)\big)^{1/2}\,\|f\|$$

and an elementary compactness argument shows that the constant may be chosen independent of ζ i.e. there is $C_A>0$ depending only on the compact set A such that

$$\big|f(\zeta)\big|\leq C_A\,\|f\| \tag{1}$$

for every $\zeta\in A$ and every $f\in L^2H(\Omega)$. The simple estimate (1) has dramatic consequences, starting with the tautology that the evaluation functional

$$\delta_\zeta: L^2H(\Omega)\to\mathbb{C},\quad \delta_\zeta(f)=f(\zeta),$$

is continuous, and ending with the deeper result that $L^2H(\Omega)$ is a closed subspace of $L^2(\Omega)$, and hence a Hilbert space itself. Therefore, by the classical Riesz theorem, the continuous functional $\delta_\zeta \in \left[L^2H(\Omega)\right]^*$ may be represented i.e. there is a unique $K(\,\cdot\,,\,\zeta) \in L^2H(\Omega)$ such that

$$f(\zeta) = \big\langle f,\ K(\,\cdot\,,\,\zeta)\big\rangle_{L^2}\,, \quad f \in L^2H(\Omega),$$

thus organizing $L^2H(\Omega)$ as a reproducing kernel Hilbert space[1] (RKHS) in the sense of N. Aronszajn[2]. The function $K : \Omega \times \Omega \to \mathbb{C}$ is the *Bergman kernel*[3] of Ω. As it turns out $K(z,\,\zeta)$ is holomorphic in the n variables z and anti-holomorphic in the last n variables ζ i.e. $K \in HA(\Omega)$. By the way $HA(\Omega)$ is a complex Fréchet space whose topology as a locally convex space is determined by the family of semi-norms

$$\big\{\|\cdot\|_A\ :\ A \subset \Omega,\ \ A \text{ compact}\big\},$$

$$\|F\|_A = \sup_{(z,\zeta)\in A\times A} \big|F(z,\,\zeta)\big|, \quad F \in HA(\Omega).$$

[We shall have the occasion to use the space $HA(\Omega)$ later on in this talk]. The notion of a reproducing kernel is however much older and was perhaps first introduced by the famous Polish mathematician S. Zaremba in connection with his work on boundary value problems for harmonic and biharmonic functions[4].

Weighted Bergman kernels are build in a quite similar manner, except that integration is performed with respect to the weighted Lebesgue measure $\gamma(z)\ d\,\mu(z)$. A *weight* is just a positive measurable function $\gamma : \Omega \to \mathbb{R}$, and the set of all weights will be denoted by $W(\Omega)$. The subset $AW(\Omega) \subset W(\Omega)$ of all *admissible* weights consists of all $\gamma \in W(\Omega)$ such that

i) $\delta_\zeta : L^2H(\Omega,\,\gamma) \to \mathbb{C}$ is continuous,

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[1] N. Aronszajn, *Theory of reproducing kernels*, Trans. Amer. Math. Soc., 68(1950), 337-404.

[2] Nachman Aronszajn (26 July 1907 – 5 February 1980) was a Polish American mathematician, and an Ashkenazi Jew. **A.** got a degree in mathematics in 1930, from the University of Warsaw, under the supervision of Stefan Mazurkiewicz, and a Ph.D. in mathematics in 1935, from Paris University, with Maurice Fréchet as an advisor. The mentioned work (cf. the previous footnote) appeared while **A.** was on the Oklahoma A & M faculty. The civil views of **A.** were not amended by his religious background, for **A.** moved to the University of Kansas in 1951 with his colleague Ainsley Diamond after Diamond, a Quaker, was fired for refusing to sign a newly instituted loyalty oath.

[3] Cf. S. Bergman, *Über die Kernfunktion eines Bereiches und ihr Verhalten am Rande*, J. Reine Angew. Math., 169(1933), 1-42.

[4] S. Zaremba, *L'equation biharmonique et une class remarquable de fonctions fondamentales harmoniques*, Bulletin International de l'Accademie des Sciences de Cracovie, 39(1907), 147-196.

ii) $L^2H(\Omega,\,\gamma)$ is a closed subspace of $L^2(\Omega,\,\gamma)$.

Therefore, if $\gamma \in AW(\Omega)$ then $L^2H(\Omega,\,\gamma)$ is a RKHS, whose unique reproducing kernel (the *weighted Bergman kernel* of Ω, of *weight* γ) is denoted by $K_\gamma(z,\,\zeta)$.

The pioneering studies on weighted Bergman kernels belong to another Polish mathematician i.e. to Z. Pasternak-Winiarski[5]. The results we wish to communicate rely strongly on those by Z. Pasternak-Winiarski, and mention[6] of those will certainly be made again later in this talk.

2. Fefferman's asymptotic expansion formula

Bergman kernels, whether weighted or not, are rather difficult to compute, and indeed they were explicitly computed only for a handful of domains $\Omega \subset \mathbb{C}^n$ e.g. for the unit ball $\Omega = \mathbb{B}^n$

$$K(z,\,\zeta) = \frac{n!}{\pi^n}\,\frac{1}{\left(1 - z\cdot\overline{\zeta}\right)^{n+1}}\,.$$

Or, to give an example we shall need shortly, if

$$\gamma_\alpha(z) = \left(\operatorname{Im}(z) - \left|z'\right|^2\right)^\alpha, \quad \alpha > -1, \quad z \in \Omega_n\,,$$

then $\gamma_\alpha \in AW(\Omega_n)$ and the corresponding weighted Bergman kernel with weight γ_α is

$$K_{\gamma_\alpha}(z,\,\zeta) = \frac{2^{n-1+\alpha} c_{n,\alpha}}{\left[i\left(\overline{\zeta}_1 - z_1\right) - 2\left\langle z'\,,\,\zeta'\right\rangle\right]^{n+1+\alpha}}\,,$$

$$c_{n,\alpha} = \pi^{-n}(\alpha+1)\cdots(\alpha+n).$$

The main ingredient in the calculation of the Bergman kernel for the ball is producing a complete orthonormal system $\{\phi_\nu\}_{\nu\geq 0}$ for the Hilbert

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[5] Z. Pasternak-Winiarski, *On weights which admit the reproducing kernel of Bergman type*, Internat. J. Math. & Math. Sci., (1)15(1992), 1-14; *On the dependence of the reproducing kernel on the weight of integration*, Journal of Functional Analysis, 94(1990), 110-134.

[6] The work by Z. Pasternak-Winiarski in Journal of Functional Analysis (cf. the previous footnote) is foundational for the theory of weighted Bergman kernels. Though some of the scientific creation of Z. Pasternak-Winiarski is not published in equally illustrious (cf. the previous footnote) mathematical journals, it is the firm belief of this speaker that insufficient credit is given to Z. Pasternak-Winiarski within the mathematical literature devoted to complex analysis.

space $L^2H(\mathbb{B}^n)$ and explicitly summing[7] the series

$$\sum_{\nu=0}^{\infty} \phi_\nu(z)\,\overline{\phi_\nu(\zeta)}\,,$$

which is known to converge (uniformly on any compact subset of $\Omega\times\Omega$) to $K(z,\,\zeta)$. The calculation of the weighted Bergman kernel $K_{\gamma_\alpha}(z,\,\zeta)$ is due to E. Barletta and S. Dragomir[8] and relies on a technique introduced into mathematical practice by M.M. Djrbashian and A.H. Karapetyan[9].

In general only asymptotic information close to the boundary may be got about (weighted) Bergman kernels. Perhaps the first result in this direction belongs to N. Kerzman[10] and establishes the differentiability up to the boundary of the Bergman kernel of a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$ i.e.

$$K \in C^\infty\big(\overline{\Omega} \times \overline{\Omega} \setminus \Delta\big),$$
$$\Delta = \big\{(z,\,\zeta) \in \partial\Omega \times \partial\Omega \;:\; z = \zeta\big\}.$$

The proof relies on the solution to the $\overline{\partial}$-Neumann problem, which at the time Kerzman's paper was written was known only for strictly pseudoconvex domains. So we should take a small step backwards and, faithful to our commitment in the Abstract to this talk, explain the strict pseudoconvexity requirement (on the boundary $\partial\Omega$) in complex analysis of functions of several complex variables.

Let $\varphi \in C^\infty(U)$ be a defining function for Ω, with $U \subset \mathbb{C}^n$ open, i.e.

i) $\overline{\Omega} \subset U$,

ii) $\Omega = \big\{z \in U \;:\; \varphi(z) < 0\big\}$, $\partial\Omega = \big\{z \in U \;:\; \varphi(z) = 0\big\}$,

iii) $\nabla\varphi(z) \neq 0$ for every $z \in \partial\Omega$.

Let L_φ be the Levi form of $\partial\Omega$ i.e.

$$L_\varphi(Z,\,\overline{W}) = -i\,(d\theta)(Z,\,\overline{W}), \quad \theta := \frac{i}{2}\big(\overline{\partial} - \partial\big)\varphi,$$

[7]Cf. e.g. S.G. Krantz, *Function theory of several complex variables*, John Wiley & Sons, New York, 1982.

[8]E. Barletta & S. Dragomir, *On the Djrbashian kernel of a Siegel domain*, Stud. Math., 127(1998), 47-63.

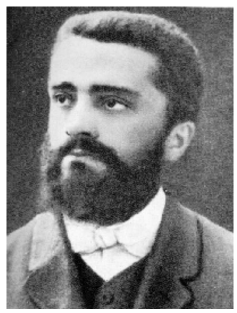
[9]M.M. Djrbashian & A.H. Karapetyan, *Integral representations for some classes of functions holomorphic in a Siegel domain*, J. Math. Anal. Appl., 179(1993), 91-109.

[10]N. Kerzman, *The Bergman kernel function. Differentiability at the boundary*, Math. Ann., 195 (1972), 149-158.

$$Z, W \in T_{1,0}(\partial\Omega) = T'\big(\mathbb{C}^n\big) \cap \big[T(\partial\Omega) \otimes \mathbb{C}\big],$$

where $T'\big(\mathbb{C}^n\big) \to \mathbb{C}^n$ is the holomorphic tangent bundle over $\mathbb{C}^n$ [the span of $\{\partial/\partial z_j : 1 \leq j \leq n\}$]. The domain Ω, or rather its boundary $\partial\Omega$, is *strictly pseudoconvex* if $L_\varphi(z)$ is positive definite at every boundary point $z \in \partial\Omega$. The Levi form is due to Eugenio Elia Levi, perhaps the best Italian mathematician ever.

Eugenio Elia Levi (1883-1917)



E.E. Levi was born on the 18th of October 1893 in Turin and died in war, shot in the head, at a location near Cormons (Gorizia) on the 28th of October 1917. His death was surely the greatest loss suffered[11] by the Italian mathematics - and not only - due to the 1914-1918 war. **L.** completed his university studies at *Scuola Normale Superiore* of Pisa in 1904 and served there as an assistant of Ulisse Dini. In 1909 **L.** became a professor of infinitesimal analysis at the University of Genova where he remained until he was called for the military service and the successive all too early ending. As F. Tricomi wrote, in spite of his premature death (when only 34) **L.** *may be considered* (on the basis of the about thirty works he wrote) *one of the major Italian mathematicians of the twentieth century.* Remarkable are **L.**'s works on second order elliptic partial differential equations (1907-1908) and also his works on the heat equation and on arguments of variational calculus. **L.** also has contributions in differential geometry and group theory. **L.** was a correspondent member of *Accademia Nazionale dei Lincei* (nominated in 1911).

A deep result on the asymptotic behavior of the Bergman kernel $K(z, \zeta)$ of a smoothly bounded strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$

[11]The biographical notes on **L.** are based on material by G. Loria and G. Fubini [Boll. Bibl. Storia Mat., (2)1(1918), 38-45] and by C.S. Roero, `http://www.torinoscienza.it/accademia/personaggi`.

is Fefferman's asymptotic development i.e.

$$(2) \qquad K(z,\, z) = C_\Omega \left|\nabla\varphi(z)\right|^2 \cdot \det L_\varphi(z) \cdot \left|\varphi(z)\right|^{-(n+1)} + E(z,\, z),$$

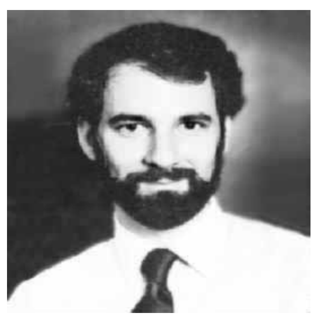
$$(3) \qquad \left|E(z,\, z)\right| \le C'_\Omega \left|\varphi(z)\right|^{-(n+1)+\frac{1}{2}} \left|\log\left|\varphi(z)\right|\right|,$$

that we only write on the diagonal of $\Omega\times\Omega$ to avoid too much notation.

An inhomogeneous audience such as the present one, may wish to know what does the asymptotic formula (2) do for you?

The first use of (2), combined with an analysis of the behavior near the boundary $\partial\Omega$ of the geodesics of the Bergman metric of Ω, was within the proof of the celebrated Fefferman theorem[12] that biholomorphisms of smoothly bounded strictly pseudoconvex domains extend smoothly at the boundary (to give a CR isomorphism there).

Charles Louis Fefferman (B. 1949)



C. Fefferman was born in Washington on the 18th of April 1949. **F.** was a child prodigy who mastered calculus before the age of twelve and entered the University of Maryland in 1966 successively graduating with the highest distinction. **F.** was awarded his Ph.D. in 1969 for his thesis *Inequalities for strongly regular convolution operators* under the supervision of Elias Stein, at the Princeton University. **F.** contributed several innovations to analysis in several complex variables by finding the appropriate generalizations of classical one complex variable results. In 1976 **F.** was awarded the Alan T. Waterman award. **F.**'s work on partial differential equations, Fourier analysis, in particular convergence, multipliers, divergence, singular integrals and Hardy spaces brought him the Fields Medal in 1978. In 1984 **F.** was

[12]C. Fefferman, *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Invent. Math., 26(1974), 1-65; *Monge-Ampére equations, the Bergman kernel, and geometry of pseudoconvex domains*, Ann. Math., 103(1976), 395-416.

appointed Herbert Jones Professor at Princeton. **F.** made striking contributions to the study of the Bergman kernel and in 1992 he was awarded the Bergman Prize.

As another elementary yet far reaching consequence of (2)-(3) together with l'Hôpital rule,

$$\rho(z) = -K(z,\, z)^{-1/(n+1)}$$

is a defining function for Ω, hence the differential 1-form θ on $\partial\Omega$ and the $(0,2)$-tensor field g_B on Ω defined by

$$\theta = \frac{i}{2}\big(\overline{\partial} - \partial\big)\rho, \quad g_{j\overline{k}} = \frac{\partial^2 \log K(z,\, z)}{\partial z_j\, \partial \overline{z}_k}\,, \tag{4}$$

are respectively a contact form on $\partial\Omega$ and a Kählerian metric on Ω (the *Bergman metric*[13] of Ω). Then (4) yields an explicitly computable relationship between the contact structure of the boundary of Ω and the Kählerian geometry of its interior. Said relationship was exploited by R. Graham and J.M. Lee[14] in their study of the C^∞ regularity up to the boundary of the solution to the Dirichlet problem

$$\Delta_B u = 0 \quad \text{in} \;\; \Omega, \qquad u = f \quad \text{on} \;\; \partial\Omega,$$

with $f \in C^\infty\big(\partial\Omega\big)$. Here Δ_B is the Bergman Laplacian i.e. the Laplace-Beltrami operator of the Riemannian manifold $(M,\, g_B)$

$$\Delta_B u = - \sum_{a=1}^{2n} \Big\{ E_a\big(E_a\, u\big) - \Big(\nabla^{g_B}_{E_a} E_a\Big)\, u \Big\}.$$

The first step in the C^∞ regularity up to the boundary problem is to look for compatibility equations

$$\mathscr{C}(f) = 0 \quad \text{along} \;\; \partial\Omega\,, \tag{5}$$

that the boundary datum f must satisfy[15]. Considering a point of the boundary $z_0 \in \partial\Omega$, and a sequence of interior points $\{z_\nu\}_{\nu \geq 1} \subset \Omega$ such that $z_\nu \to z_0$ for $\nu \to \infty$, and merely looking at

$$\lim_{\nu \to \infty} \big(\Delta_B\, u\big)(z_\nu)$$

[13]Remarkably, the Bergman metric springs solely from the complex structure of Ω [and biholomorphisms of Ω are isometries of g_B].

[14]C.R. Graham & J.M. Lee, *Smooth solutions of degenerate Laplacians on strictly pseudoconvex domains*, Duke Math. J., (3)57(1988), 697-720.

[15]The next, and more ambitious, step is of course to see whether the compatibility conditions $\mathscr{C}(f) = 0$ that one discovered suffice for proving that the Dirichlet problem for the Bergman Laplacian possesses a solution $u \in C^\infty\big(\overline{\Omega}\big)$.

isn't going to work because Δ_B is built in terms of the Levi-Civita connection ∇^B, whose Christoffel coefficients do not stay bounded at the boundary. A more subtle asymptotic analysis as $\partial\Omega$ is approached is needed, and that can be achieved as follows.

Famous work by G.R. Graham and J.M. Lee[16] builds a linear connection ∇ (the *Graham-Lee connection*) on a onesided neighborhood of the boundary $U \subset \overline{\Omega}$, that is foliated by level sets

$$M_\epsilon = \{z \in \overline{\Omega} \;:\; \varphi(z) = -\epsilon\}, \quad 0 \le \epsilon \le \epsilon_0 \,,$$

[with ϵ_0 sufficiently small, so that each M_ϵ inherits strict pseudoconvexity from $M_0 = \partial\Omega$] whose pointwise restriction to a leaf M_ϵ is the Tanaka-Webster connection of that leaf, familiar within S.M. Webster's pseudohermitian geometry on each M_ϵ. The magnificent role played in this story by Fefferman's asymptotic expansion of the Bergman kernel is that it allows expressing the Levi-Civita connection ∇^B of the Bergman metric g_B in terms of the Graham-Lee connection ∇ and the transverse curvature[17]

$$r = 2\,\partial\overline{\partial}\varphi\,(\xi,\ \overline{\xi})$$

[of the foliation $\mathscr{F}$ of U by level sets of $\varphi(z) = -K(z,\,z)^{-1/(n+1)}$]. Then the restriction to M_ϵ of the equation $\Delta_B u = 0$ yields

$$\mathscr{C}_\epsilon(u \circ j_\epsilon) = 0 \quad \text{along} \quad M_\epsilon\,, \tag{6}$$

$$j_\epsilon : M_\epsilon \hookrightarrow \Omega, \quad 0 < \epsilon \le \epsilon_0\,,$$

and then an elementary asymptotic analysis as $\epsilon \to 0$ will eventually lead to the compatibility equations (5) simply because the Graham-Lee connection ∇ stays bounded at the boundary $\partial\Omega$ (the pointwise restriction to M_ϵ of ∇ is a pseudohermitian invariant of M_ϵ, and each M_ϵ has the same pseudohermitian geometry as $M_0 = \partial\Omega$) and (by a result of J.M. Lee and B. Melrose[18]) the transverse curvature of $\mathscr{F}$ is smooth up to the boundary

$$r \in C^\infty\big(\overline{\Omega} \cap U\big).$$

It is interesting to remark that (6) is a polynomial in the "indeterminate" $1/\varphi$ and that as $\epsilon \to 0$ only the coefficients of the terms of

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[16]Cf. *op. cit.*

[17]Here ξ is the complex vector field on U of type $(1,0)$ determined by

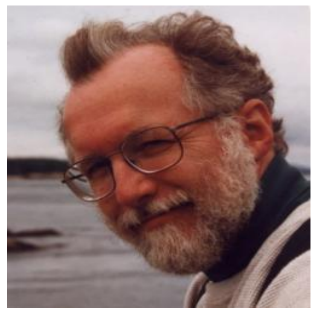
i) $\partial\varphi(\xi) = 1$,

ii) ξ is orthogonal to $T_{1,0}(\mathcal{F})$ with respect to $\partial\overline{\partial}\varphi$ $\big[$i.e. $\partial\overline{\partial}\varphi(\xi, \overline{Z}) = 0$ for any $Z \in T_{1,0}(\mathcal{F})$.$\big]$

[18]J. Lee & R. Melrose, *Boundary behavior of the complex Monge-Ampère equation*, Acta Math. 148(1982),159-192.

order $O\big(\varphi^{-N}\big)$ will survive to give (5), where N is the degree of that polynomial.

John M. Lee (B. 1950)



J.M. Lee was born on the 2nd of September 1950. **L.** studied at Princeton University (1968-1972) and Tufts University (1977-1978) and was awarded his Ph.D. in 1982 at Massachusetts Institute of Technology for his thesis *Higher asymptotics of the complex Monge-Ampère equation and the geometry of CR manifolds* under the supervision of Richard B. Melrose. **L.** was appointed Professor of Mathematics at the University of Washington in 1996. **L.** has excellent contributions (together with D. Jerison) in the early applications of subelliptic theory to CR geometry (the solution to the Yamabe problem for the Fefferman metric i.e. the so called *CR Yamabe problem*. A remarkable contribution is brought by **L.** (together with C.R. Graham) to the C^∞ regularity up to the boundary in the Dirichlet problem for Bergman type Laplacians on a strictly pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Here **L.** devised a strong differential geometric investigation tool namely the *Graham-Lee connection* i.e. a linear connection on a sufficiently small one-sided neighborhood U of $\partial\Omega$ inducing the Tanaka-Webster connection on each level set contained in U of the defining function of Ω. It should also be mentioned that **L.**'s description of the Fefferman metric in terms of pseudohermitian invariants is a positive solution to a problem posed by C. Fefferman in 1976.

Other applications of Fefferman's asymptotic expansion of the Bergman kernel abound, yet the above will certainly suffice to the present audience as an explanation for the popularity enjoyed by said expansion within the complex analysis community, and for the need felt by that community to generalize the expansion to the case of weighted Bergman kernels $K_\gamma(z,\,\zeta)$.

The first generalization of the sort is the Forelli-Rudin-Ligocka-Peloso formula[19]. One may show that positive integer powers of the defining function are admissible weights i.e.

$$\big|\varphi\big|^m \in AW(\Omega), \quad m \in \mathbb{Z}_+ .$$

Let

$$K_m(z,\zeta) = K_{|\varphi|^m}(z,\,\zeta)$$

be the weighted Bergman kernel of Ω, corresponding to the weight $\gamma(z) = |\varphi(z)|^m$. Then

$$K_m(\zeta,\,z) = C_\Omega\,\big|\nabla\varphi(z)\big|^2 \times \tag{7}$$

$$\times\ \det L_\varphi(z)\ \cdot\ \Gamma(\zeta\,,z)^{-(n+1+m)} + E(\zeta,\,z),$$

$$E \in C^\infty\big(\overline{\Omega}\times\overline{\Omega}\setminus\Delta\big), \quad \Delta = \{(z,\,z)\ :\ z\in\partial\Omega\},$$

$$\big|E(\zeta,\,z)\big| \le C'_\Omega\,\big|\Gamma(\zeta,\,z)\big|^{-(n+1+m)+1/2}\big|\log|\Gamma(\zeta,\,z)|\big|. \tag{8}$$

$$F(\zeta,\,z) = -\sum_{j=1}^{n}\frac{\partial\varphi}{\partial z_j}(z)\big(\zeta_j - z_j\big) +$$

$$-\sum_{j,k=1}^{n}\frac{\partial^2\varphi}{\partial z_j\,\partial z_k}(z)\big(\zeta_j - z_j\big)\big(\zeta_k - z_k\big),$$

$$\Gamma(\zeta,\,z) = \big[F(\zeta\,,\,z) - \varphi(z)\big]\,\chi\big(|\zeta - z|\big) +$$

$$+\big[1 - \chi\big(|\zeta - z|\big)\big]\,|\zeta - z|^2\,,$$

where χ is a C^∞ cut-off function of the real variable t, such that $\chi(t) = 1$ for $|t| < \epsilon_0/2$ and $\chi(t) = 0$ for $|t| \ge 3\epsilon_0/4$.

Formula (7) [together with the estimate (8)] was proved by M.M. Peloso[20]. M.M. Peloso claims[21] that Theorem (7)-(8) is implicit in the work by E. Ligocka[22] while E. Ligocka does employ an older idea by F. Forelli & W. Rudin[23]. Aside from the correct credit, which certainly goes to M.M. Peloso, the history of Theorem (7)-(8) demonstrates the attention shown by the mathematical community (devoted to complex analysis) to an argument born with the celebrated work by C.

[19]M.M. Peloso, *Sobolev regularity of the weighted Bergman projections and estimates for minimal solutions to the $\overline{\partial}$-equation*, Complex Variables Theory Appl., 27(1995), 339-363.

[20]Cf. *op. cit.*

[21]Cf. *op. cit.*

[22]E. Ligocka, *On the Forelli-Rudin construction and weighted Bergman projections*, Stud. Math., 94(1989), 257-272.

[23]F. Forelli & W. Rudin, *Projections on spaces of holomorphic functions in balls*, Indiana Univ. Math. J., 24(1974), 593-602.

Fefferman[24] and emphasizes the recognition of the relevance of that argument.

The natural problem arrises whether the Forelli-Rudin-Ligocka-Peloso expansion can be recovered for weighted Bergman kernels $K_\gamma(z,\,\zeta)$ corresponding to ampler classes of admissible weights (other than positive integer powers of the defining function)

$$\gamma \in AW(\Omega) \setminus \{|\varphi|^m \;:\; m \in \mathbb{Z}_+\}.$$

A result in this direction is due to E. Barletta and S. Dragomir[25], yet certainly the credit should go entirely to Z. Pasternak-Winiarski[26] for his proof of the analyticity of the weighted Bergman kernel, as a function of the weight

$$K : AW(\Omega) \to HA(\Omega), \quad \gamma \longmapsto K_\gamma\,.$$

Telling the story of the analyticity of the map $\gamma \to K_\gamma$, and of its application [towards a Fefferman-type asymptotic expansion of $K_\gamma(z,\,\zeta)$] as observed by E. Barletta and S. Dragomir, will get us really close to the original purpose of this talk.

To make sense of said analyticity Z. Pasternak-Winiarski organized $W(\Omega)$ as a Banach manifold modeled on $L^\infty(\Omega,\,\mathbb{R})$, the Banach space of all real valued essentially bounded functions $g : \Omega \to \mathbb{R}$, with the norm

$$\|g\|_\infty = \operatorname{ess\,sup}_{z\in\Omega}|g(z)| =$$
$$= \inf\{K > 0 \;:\; |g(z)| \le K \;\text{ for a.e. }\; z \in \Omega\}.$$

such that $AW(\Omega)$ is an open subset in $W(\Omega)$. A first observation, once $W(\Omega)$ was assigned a topology, is that the curve

$$C : (-1,\,+\infty) \to AW(\Omega), \quad C(\alpha) = |\varphi|^\alpha\,, \quad \alpha > -1,$$

is discontinuous and actually every point of C is isolated! Taking note of the fact that a Fefferman-type expansion for $K_{|\varphi|^\alpha}(z,\,\zeta)$ is actually known only for weights that are points of C corresponding to non-negative integer values of the parameter $\alpha \in \mathbb{Z}_+$, one gets an idea of the amount of the job still undone.

The main idea in the work by E. Barletta and S. Dragomir[27] is to expand K_γ in a neighborhood $\gamma \in U(\Omega,\,|\varphi|^m)$ of a point $C(m) = |\varphi|^m$

[24] Cf. *op. cit.*

[25] E. Barletta & S. Dragomir, *On boundary behavior of symplectomorphisms*, Kodai Math. J., 21(1998), 285-305.

[26] Cf. *op. cit.*

[27] Cf. *op. cit.*

[at which the Fefferman-type asymptotic expansion is (by the Forelli-Rudin-Ligocka-Peloso formula) is known], by profiting from the fact that the map $K : AW(\Omega) \to HA(\Omega)$ is analytic[28]. Here, for every $m \in \mathbb{Z}_+$

$$U\big(\Omega\,,\,|\varphi|^m\big) = \big\{g\,|\varphi|^m \;:\; g \in U(\Omega)\big\},$$

$$U(\Omega) = \big\{g \in L^\infty(\Omega),\;\; \operatorname{ess\,inf}_{z\in\Omega}\, g(z) > 0\big\},$$

$$\operatorname{ess\,inf}_{z\in\Omega} g(z) = \sup\big\{L \in \mathbb{R} \;:\; L \le g(z) \;\text{ for a.e. }\; z \in \Omega\big\},$$

and the analytic expansion of K (as got by Z. Pasternak-Winiarski[29]) is

$$K_{(g+h)\,\gamma} = K_{g\,\gamma} + \sum_{k=1}^{\infty} (-1)^k\, K^{(k)}_{g,\,\gamma}\, h^{(k)}\,, \tag{9}$$

$$\gamma \in AW(\Omega), \quad g \in U(\Omega), \quad h \in B_g\,,$$

$$B_g = B_{i(g)/2}(0) = \Big\{h \in L^\infty(\Omega) \;:\; \|h\|_\infty < \frac{i(g)}{2}\Big\},$$

$$i(g) = \operatorname{ess\,inf}_{z\in\Omega}\, g(z).$$

Details about the construction of the k-linear maps

$$K^{(k)}_{g,\,\gamma} : L^\infty(\Omega)^k \to HA(\Omega)$$

will be given as we shall get to our main applications.

3. Transition probability amplitudes

For every admissible weight $\gamma \in AW(\Omega_n)$ let

$$\mathscr{K}_\gamma : \Omega_n \to \mathbb{CP}\big(\mathscr{H}_\gamma\big), \quad \mathscr{H}_\gamma = L^2H\big(\Omega_n\,,\,\gamma\big),$$

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[28]Let $\mathscr{X}$ and $\mathscr{Y}$ be respectively a normed space and a topological space. Let $U \subset \mathscr{X}$ be an open set. A function $F : U \subset \mathscr{X} \to \mathscr{Y}$ is *analytic* on U if for any $x \in U$ there is a ball $B \subset \mathscr{X}$ of center ζ with $x + B \subset U$, and there is a sequence $\{a_m\}_{m\in\mathbb{N}}$ of continuous multi-linear (m-linear) maps $a_m : \mathscr{X}^m \to \mathscr{Y}$ such that

$$F(x+h) = F(x) + \sum_{m=1}^{\infty} a_m\big(h,\,\cdots,\,h\big)$$

for any $h \in B$ and the series $\sum_{m\ge 1} a_m\big(h,\,\cdots,\,h\big)$ converges uniformly in $h \in B$.

[29]Cf. *op. cit.*

be the coherent state map built[30] from the Hermitian line bundle $\big(E,\ H_\gamma\big)$

$$E = \Omega_n \times \mathbb{C}, \quad H_\gamma(\sigma_0\,,\,\sigma_0) = \gamma\,,$$

and turning out[31] to be

$$\mathscr{K}_\gamma(z) = \Big[\frac{K_\gamma\big(\,\cdot\,,\,z\big)}{K_\gamma\big(z\,,\,z\big)^{1/2}}\Big] \in \mathbb{CP}\big(\mathscr{H}_\gamma\big)\,, \quad z \in \Omega_n\,.$$

Previous to A. Odijewicz's approach to $\mathscr{K}_\gamma$, coherent state map were built in terms of holomorphic sections of the given Hermitian line bundle E that are L^2 with respect to the Liouville measure. A. Odzijewicz's construction has the advantage to reduce the calculation of the transition probability amplitude $a_{0\overline{0}}(\zeta,\,z)$ from $\mathscr{K}_\gamma(z)$ to $\mathscr{K}_\gamma(\zeta)$ to the calculation of the reproducing kernel $K_\gamma(z,\,\zeta)$ i.e.

$$a_{0\overline{0}}(\zeta,\,z) = \frac{K_\gamma\big(\zeta,\,z\big)}{K_\gamma(z,\,z)^{1/2}\,K_\gamma\big(\zeta,\,\zeta\big)^{1/2}}\,.$$

As observed by A. Odzijewicz himself this brings in other difficulties, related to the need to recover $a_{0\overline{0}}(\zeta,\,z)$ by averaging[32] over $w \in \Omega$ in

$$a_{0\overline{0}}(w,\,z)\,a_{0\overline{0}}(\zeta,\,w)$$

(the transition probability amplitude from z to ζ with *simultaneous transition* though w). A. Odzijewicz choses to circumnavigate said difficulty by assuming that the measure on phase space [associated to the

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[30]Cf. A. Odzijewicz, *On reproducing kernels and quantization of states*, Commun. Math. Phys., 114(1988), 577-597. Here $\sigma_0 : \Omega_n \to E$ is the global section in $E = \Omega_n \times \mathbb{C}$ given by $\sigma_0(z) = (z,\ 1)$, $z \in \Omega_n$. The original construction of the coherent state map is more general and springs from an arbitrary Hermitian line bundle, on which a holomorphic trivialization atlas has been fixed. When the base manifold of the given holomorphic line bundle is a domain $\Omega \subset \mathbb{C}^n$, the bundle itself is the trivial line bundle $E = \Omega \times \mathbb{C}$, and the function $\gamma = H(\sigma_0\,,\,\sigma_0)$ [determining the given Hermitian metric H on E] is an admissible weight [i.e. $\gamma \in AW(\Omega)$] A. Odzijewicz's construction (cf. *op. cit.*) was recognized by Z. Pasternak-Winiarski (cf. *op. cit.*) to lead to the map $\mathscr{K}_\gamma$.

[31]Cf. Z. Pasternak-Winiarski, *op. cit.*, and also E. Barletta & S. Dragomir & F. Esposito, *Weighted Bergman Kernels and Mathematical Physics*, Axioms, 9(2020), 1-48, `doi:10.3390`.

[32]With respect to the Liouville measure

$$d\,\mu_L = (-i)^n\,\Omega_{0\overline{0}}\big(\gamma\big)\,d\zeta^1 \wedge \cdots \wedge d\zeta^n \wedge d\overline{\zeta}^1 \wedge \cdots \wedge d\overline{\zeta}^n\,,$$

$$\Omega_{0\overline{0}}\big(\gamma\big) := \det\big[\omega_{j\overline{k}}\big]\,, \quad \omega = \operatorname{curv}\big(E,\,H_\gamma\big).$$

reproducing kernel of $L^2H(\Omega,\,\gamma)$] should coincide, up to a multiplicative constant, with the Liouville measure. As successively observed[33] A. Odzijewicz's assumption is unnecessarily strong[34], and can be refuted by looking at the Siegel domain $\Omega = \Omega_n$ (and the trivial vector bundle over Ω_n, equipped with the Hermitian metric determined by the admissible weight γ_α). This is the reason the remaining part of this talk is confined to the Siegel domain, and then $a_{0\overline{0}}(z,\,\zeta)$ is readily computable from our explicit knowledge of $K_{\gamma_\alpha}(z,\,\zeta)$ i.e.

$$a_{0\overline{0}}\big(\zeta,\,z\big) = \left[\frac{2\,\rho(z)^{\frac{1}{2}}\,\rho(\zeta)^{\frac{1}{2}}}{i\big(\overline{z}_1 - \zeta_1\big) - 2\langle\zeta'\,,\,z'\rangle}\right]^{n+\alpha+1},$$

$$\rho(z) = \mathrm{Im}(z_n) - \big|z'\big|^2.$$

However if the mechanical system (whose classical phase space is Ω_n) interacts (at the quantum level) with an external field[35] $B : \Omega_n \to \mathbb{R}$ then the coherent states change

$$\mathscr{K}_{\alpha,\,B} : \Omega_n \to \mathbb{CP}\big(L^2H(\Omega_n\,,\,e^B\,\gamma_\alpha)\big)$$

and the problem of computability of the transition probability amplitude from $\mathscr{K}_{\alpha,\,B}(z)$ to $\mathscr{K}_{\alpha,\,B}(\zeta)$ goes back to the computability of the weighted Bergman kernel

$$K_{e^B\,\gamma_\alpha}\big(z,\;\zeta\big) \tag{10}$$

corresponding to the (admissible) weight $e^B\,\gamma_\alpha \in AW\big(\Omega_n\big)$. And the explicit calculation of the reproducing kernel (10) appears formidable!

As successful veterans of generalizing Fefferman's asymptotic formula for the weighted Bergman kernel $K_\gamma(z,\,\zeta)$ to ampler classes of weights, by using the analyticity of the map $\gamma \in AW(\Omega) \mapsto K_\gamma \in HA(\Omega)$ and Z. Pasternak-Winiarski's expansion of K_γ (on which the proof of said analyticity relies) we attempt the calculation of $K_{e^B\,\gamma_\alpha}$ by

[33]Cf. E. Barletta & S. Dragomir & F. Esposito, *Kostant-Souriau-Odzijewicz quantization of a mechanical system whose classical phase space is a Siegel domain*, Internat. J. Reproducing Kernels, (1)1(2022), 1-19.

[34]The proportionality constant among the measure determined by the reproducing kernel of $L^2H(\Omega,\,\gamma)$ and the Liouville measure needs not be independent of the weight γ.

[35]That external fields may be mathematically described as deformations of the holomorphic and Hermitian structures of the given line bundle $E \to \Omega_n$ is an idea going back to R. Penrose [cf. R. Penrose & M.A.H. MacCallum, *Twistor theory: an approach to the quantization of fields and space-time*, Phys. Rep., (4)6(1972), 241-316]. For simplicity, we keep the holomorphic structure fixed and only deform the Hermitian metric $H_{\gamma_\alpha} \mapsto e^B\,H_{\gamma_\alpha}$.

building once again on the results by Z. Pasternak-Winiarski, cf. *op. cit.*, i.e. for any $g \in U(\Omega_n)$ and any $h \in B_{i(g)/2}(0)$

$$K_{(g+h)\gamma_a} = K_{g\gamma_\alpha} + \sum_{m=1}^{\infty} (-1)^m \, K^{(m)}_{g,\,\gamma_\alpha}\big(h, \, \cdots , \, h\big), \tag{11}$$

$$\begin{aligned} K^{(m)}_{g,\,\gamma_\alpha}\big(h_1 \, , \, \cdots , \, h_m\big)(\zeta, \, z) = \\ = \int_{\Omega_n} K_{g\gamma_\alpha}\big(w_1 \, , \, z\big) \, h_1\big(w_1\big) \, \gamma_\alpha\big(w_1\big) \, d\mu(w_1) \cdot \\ \cdot \int_{\Omega_n} K_{g\gamma_\alpha}\big(w_2 \, , \, w_1\big) \, h_2\big(w_2\big) \, \gamma_\alpha\big(w_2\big) \, d\mu(w_2) \cdot \\ \vdots \\ \cdot \int_{\Omega_n} K_{g\gamma_\alpha}\big(w_{m-1} \, , \, w_{m-2}\big) h_{m-1}\big(w_{m-1}\big) \, \gamma_\alpha\big(w_{m-1}\big) \, d\mu(w_{m-1}) \cdot \\ \cdot \int_{\Omega_n} K_{g\gamma_\alpha}\big(w_m \, , \, w_{m-1}\big) \, h_m\big(w_m\big) \, K_{g\gamma_\alpha}\big(\zeta, \, w_m\big) \, \gamma_\alpha\big(w_m\big) \, d\mu(w_m), \end{aligned} \tag{12}$$

for any $h_1 \, , \, \cdots , \, h_m \in L^\infty(\Omega_n)$. Here

$$i(g) = \operatorname{ess\,inf}_{z \in \Omega_n} g(z)$$

and $B_r(0)$ is the ball of radius $r > 0$ and center 0 in $L^\infty(\Omega_n)$. In particular

$$\begin{aligned} K^{(1)}_{g,\,\gamma_\alpha}(h)(\zeta, \, z) = \\ = \int_{\Omega_n} K_{g\gamma_\alpha}\big(w, \, z\big) \, h\big(w\big) \, K_{g\gamma_\alpha}\big(\zeta, \, w\big) \, \gamma_\alpha(w) \, d\mu(w). \end{aligned}$$

Precisely, we wish to exploit (11) for $g \equiv 1$ [implying $i(g) = 1$] and $1 + h = e^B$, for in that case the right hand side of (11) is expressed in terms of the kernel $K_{\gamma_\alpha}(z, \, \zeta)$ whose explicit expression is available. However the calculation of the multiple integral in the right hand side of (12) appears as an equally formidable task!

As suggested by A. Odzijewicz (cf. *op. cit.*) we only attempt said calculation for weak external fields $\epsilon \, B$, $0 < \epsilon << 1$ i.e. we use (11) for $g \equiv 1$ and

$$1 + h = e^{\epsilon \, B} \, , \quad \big\| e^{\epsilon \, B} - 1 \big\|_\infty < \frac{1}{2} \, ,$$

for ϵ sufficiently small[36], and then compute the kernel and the transition probability amplitudes [from the coherent state $\mathscr{K}_{\alpha, \, \epsilon \, B}(z)$ to the

[36] A small piece of mathematical analyis shows that a necessary and sufficient condition for the existence of $\epsilon_0 > 0$ such that $\big\| e^{\epsilon \, B} - 1 \big\|_\infty < \frac{1}{2}$ for any $0 < \epsilon < \epsilon_0$, is that $B \in L^\infty(\Omega_n)$.

coherent state $\mathscr{K}_{\alpha,\,\epsilon\,B}(\zeta)]$ only to order $O(\epsilon^2)$ i.e.

$$a_{0\bar{0}}(\zeta,\,z) = \Big[\frac{2\,\rho(z)^{1/2}\rho(\zeta)^{1/2}}{i(\overline{z}_1-\zeta_1)-2\langle\zeta'\,,\,z'\rangle}\Big]^{n+1+\alpha}+ \\ +\epsilon\,G(\zeta,z)+\mathrm{O}(\epsilon^2) \tag{13}$$

where the $\mathrm{O}(\epsilon)$ term is given by

$$G(\zeta,\,z) = \frac{8}{(c_{n,\alpha})^2}\,\rho(\zeta)^{(n+1+\alpha)/2}\,\rho(z)^{(n+1+\alpha)/2}\,\times \\ \times\int_{\Omega_n}\Big\{\Big[\rho(z)^{n+1+\alpha}\Big|K_{\gamma_a}\big(w,\,z\big)\Big|^2+ \\ +\rho(\zeta)^{n+1+\alpha}\Big|K_{\gamma_\alpha}\big(w,\,\zeta\big)\Big|^2\Big]K_{\gamma_\alpha}\big(\zeta,\,z\big)+ \\ -\frac{c_{n,\alpha}}{2}\,K_{\gamma_\alpha}\big(w,\,z\big)\,K_{\gamma_\alpha}\big(\zeta,\,w\big)\Big\}\,B(w)\,\rho(w)^\alpha\,d\mu(w). \tag{14}$$