Second order q-difference equations solvable by factorization method

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In memory of Prof. Anatol Odzijewicz

The main purpose of this presentation is to apply the factorization method to difference equations. The factorization method offers the possibility of finding solutions of new classes of difference equations.

The factorization is a well-known method of solving differential equations, having roots in the works of G. Darboux, E. Schrödinger, P.A.M. Dirac, L. Infeld and T.E. Hull, W.Jr. Miller, B. Mielnik. Later this method (and some its modification) was applied to study of the most important eigenvalue problems in quantum mechanics for well-known solvable potentials like: harmonic oscillator, isotropic oscillator, Morse, Rosen-Morse, Eckart, Pöschl-Teller, etc. Additionally, the modified factorization method was application to the other topics as: supersymmetric quantum mechanics, shape–invariant potentials, inverse scattering method, coherent states, etc.

The basic idea of the factorization method is well known. One says that operators $\mathbf{H}_k$ admit factorization if there exist sequences of operators $\mathbf{A}_k$, $\mathbf{A}_k^*$ and constants a_k such that

$$\boxed{\mathbf{H}_k = \mathbf{A}_k^*\mathbf{A}_k + a_k = \mathbf{A}_{k+1}\mathbf{A}_{k+1}^* + a_{k+1} \qquad \text{for} \qquad k \in \mathbb{N} \cup \{0\}\ .}$$

Eigenproblem for the chain of operators

If the operators $\mathbf{H}_k$ admit the factorization then the eigenproblem

$$\boxed{\mathbf{H}_k \psi_k^n = \lambda_k^n \psi_k^n}$$

can be rewritten in one of the following forms

$$\mathbf{A}_k^* \mathbf{A}_k \psi_k^n = (\lambda_k^n - a_k)\psi_k^n \ , \tag{1}$$

$$\mathbf{A}_{k+1} \mathbf{A}_{k+1}^* \psi_k^n = (\lambda_k^n - a_{k+1})\psi_k^n \ . \tag{2}$$

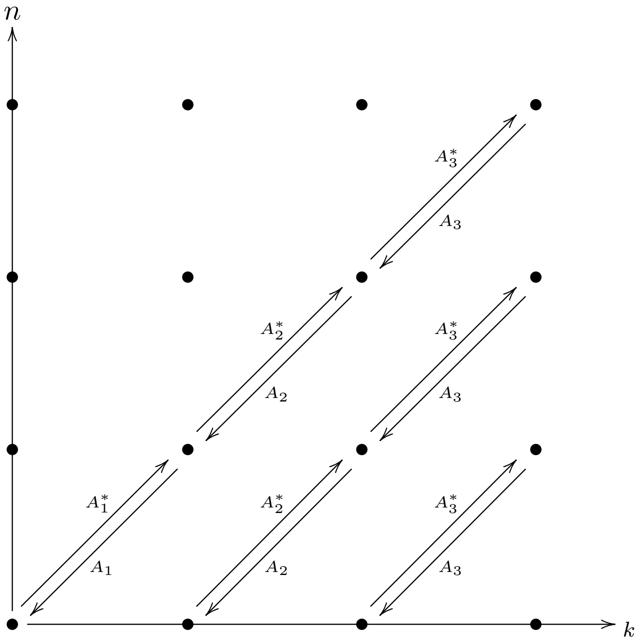
There exist some important classes of factorizations with the property $\lambda_k^0 = a_k$. In these cases any nonzero solution of the equation (ground state)

$$\mathbf{A}_k \psi_k^0 = 0$$

is automatically a solution of eigenproblem . Moreover (1) and (2) can be used to construct new solutions

$$\psi_k^n(x) := \mathbf{A}_k^* \dots \mathbf{A}_{k-n+1}^* \psi_{k-n}^0(x),$$

with eigenvalues $\lambda_k^n = a_{k-n}$ for any $n \in \mathbb{N}$.


n
k
A_1^*
A_1
A_2^*
A_2
A_2^*
A_2
A_3^*
A_3
A_3^*
A_3
A_3^*
A_3

The annihilation and creation operators are of the form

$$\mathbf{A}_k = \frac{d}{dx} + f_k, \qquad \mathbf{A}_k^* = -\frac{d}{dx} + f_k$$

and the operators $\boxed{\mathbf{H}_k = -\frac{d^2}{dx^2} + V_k(x).}$
The factorization condition

$$\mathbf{A}_k^*\mathbf{A}_k + a_k = \mathbf{A}_{k+1}\mathbf{A}_{k+1}^* + a_{k+1}$$

is equivalent to the system of non–linear differential equations

$$\boxed{f_{k+1}^2 - f_k^2 + f_{k+1}' + f_k' = a_k - a_{k+1}, \quad k \in \mathbb{N} \cup \{0\}}$$

The equation was considered in many papers, but nevertheless for these differential–difference equations there is no complete theory. One of the methods for solving is to look for the solutions in the form of infinite series $f_k(x) = \sum_{i\in\mathbb{Z}} \tilde{f}_i(x)k^i$ and obtain in this way the conditions on the function $\tilde{f}_i(x)$. The case of solutions given by the finite series were consider by Infeld and Hull. The classification of all factorisable one–dimensional problems is still an open

We consider the sequence of the eigenvalue problems for the second order q–difference operators

$$\mathbf{H}_k\psi_k(x) = \left(Z_k(x)\partial_q Q^{-1}\partial_q + W_k(x)\partial_q + V_k(x)\right)\psi_k(x) =$$

$$= \lambda_k\psi_k(x),$$

$$k \in \mathbb{N} \cup \{0\}.$$

Z_k, W_k, V_k are real-valued functions.

$$\partial_q\psi(x) = \frac{\psi(x)-\psi(qx)}{(1-q)x}$$ **q–derivative operator**

$$Q^{\pm 1}\psi(x) = \psi(q^{\pm 1}x)$$ **shift operator**

$$0 < q < 1$$

In this presentation we deal with a factorization method applied to the q–difference operators

$$\mathbf{H}_k = Z_k(x)\partial_q Q^{-1}\partial_q + W_k(x)\partial_q + V_k(x)\ .$$

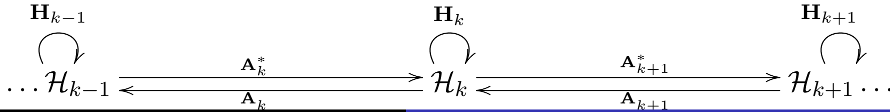
One says that operators $\mathbf{H}_k$ admit factorization if there exist sequences of first order q–difference operators $\mathbf{A}_k$, $\mathbf{A}_k^*$ and constants a_k, d_k such that

$$\boxed{\mathbf{H}_k = \mathbf{A}_k^*\mathbf{A}_k + a_k = d_{k+1}^{-1}\left(\mathbf{A}_{k+1}\mathbf{A}_{k+1}^* + a_{k+1}\right)\ ,}$$

for

$$\mathbf{A}_k : \mathcal{H}_k \to \mathcal{H}_{k-1},\quad \mathbf{A}_k = \partial_q + f_k\ ,$$
$$\mathbf{A}_k^* : \mathcal{H}_{k-1} \to \mathcal{H}_k,\quad \mathbf{A}_k^* = (\partial_q + f_k)^* = -A_k - B_k\partial_q Q^{-1} + f_k\eta_k.$$

Adjoint operator $\langle \mathbf{A}_k^*\psi_{k-1}|\varphi_k\rangle_k = \langle \psi_{k-1}|\mathbf{A}_k\varphi_k\rangle_{k-1}$.



The operators H_k acting in the Hilbert spaces $\mathcal{H}_k$. By definition the Hilbert spaces

$$\mathcal{H}_k = L^2\left([a,b]_q, \varrho_k d_q x\right)$$

consist of the complex valued functions $\psi : [a,b]_q \longrightarrow \mathbb{C}$ defined on the q–interval

$$[a,b]_q = \{q^n a : n \in \mathbb{N} \cup \{0\}\} \cup \{q^n b : n \in \mathbb{N} \cup \{0\}\}$$

and square-integrable with respect to the scalar products

$$\langle \psi | \varphi \rangle_k := \int_a^b \overline{\psi(x)} \varphi(x) \varrho_k(x) d_q x$$

defined by Jackson q–integral
$\int_0^a \psi(x) d_q x := \sum_{n=0}^{\infty} (1-q) q^n a \psi(q^n a)$.

• We postulate following recurrence relations between weight functions:

$$\varrho_{k-1} = \eta_k \varrho_k \quad ,$$

$$\varrho_{k-1} = Q\left(B_k \varrho_k\right) \quad ,$$

where η_k, B_k are real valued functions.

• The functions B_k, η_k satisfy the equation

$$Q\left(B_k \varrho_k\right) = \eta_k \varrho_k \quad . \tag{3}$$

• Additionally we put a boundary condition

$$B_k(a)\varrho_k(a) = B_k(b)\varrho_k(b) = 0 \quad .$$

After introducing the function:

$$A_k(x) = \frac{B_k(x) - \eta_k(x)}{(1-q)x}$$

we obtain from (3) the q–Pearson equation

$$\partial_q\left(B_k \varrho_k\right) = A_k \varrho_k \quad .$$

The conditions

$$\mathbf{H}_k = \mathbf{A}_k^* \mathbf{A}_k + a_k = d_{k+1}^{-1} \left(\mathbf{A}_{k+1} \mathbf{A}_{k+1}^* + a_{k+1} \right) \ ,$$

are equivalent to the following equations

1. $$\eta_{k+1}(x) = g_k(x) \eta_k(q^{-1} x),$$

2. $$\varphi_{k+1}(x) = \frac{d_{k+1}}{g_k(x)} \varphi_k(q^{-1} x) \ ,$$

3. $$\varphi_k^2(x) \eta_k(x) - \frac{g_k(qx)}{d_{k+1}} \varphi_k^2(qx) \eta_k(qx) =$$

$$= \left(\frac{q^2 d_{k+1} B_k(qx) - g_k(q^2 x) B_k(q^2 x)}{(1-q)^2 q^3 x^2} + d_{k+1} a_k - a_{k+1} \right) \frac{g_k(qx)}{d_{k+1}^2} \ ,$$

where

$$g_k(x) := \frac{B_{k+1}(x)}{B_k(x)} \ , \quad \varphi_k(x) := f_k(x) + \frac{1}{(1-q)x} \ .$$

Formulas 1 and 2 give the transformation rules equivalent to

$$\eta_k(x) = g_{k-1}(x) g_{k-2}(qx) \dots g_0(q^{-k+1}x) \eta_0(q^{-k}x) \quad ,$$

$$B_k(x) = g_{k-1}(x) g_{k-2}(x) \dots g_0(x) B_0(x) \quad ,$$

$$\varphi_k(x) = \frac{d_k \dots d_1}{g_{k-1}(x) \dots g_0(q^{-k+1}x)} \varphi_0(q^{-k}x) \quad .$$

Let us define the function $\alpha_k(x) := \varphi_k^2(x)\eta_k(x)$, and insert it into equations 3 to obtain

$$\alpha_0(x) - \frac{g_k(q^{k+1}x)}{d_{k+1}} \frac{G_k(x)}{G_k(qx)} \alpha_0(qx) = \frac{g_k(q^{k+1}x))G_k(x)}{d_{k+1}^2} (d_{k+1}a_k - a_{k+1} +$$

$$\frac{q^2 d_{k+1} g_{k-1}(q^{k+1}x) \dots g_0(q^{k+1}x) B_0(q^{k+1}x) - g_k(q^{k+2}x) \dots g_0(q^{k+2}x) B_0(q^{k+2}x}{(1-q)^2 q^{2k+3} x^2}$$

for $k \in \mathbf{N} \cup \{0\}$, where $G_k(x) := \frac{g_{k-1}(q^k x) \dots g_0(qx)}{(d_k \dots d_1)^2}$ for $k \in \mathbf{N}$, $G_0(x) \equiv 1$.

This is infinite sequence of equations on one function α_0. We postulate that it reduces in fact to one equation. This requirement is equivalent to the following

1. $$\frac{g_k(q^{k+1}x)}{d_{k+1}}\frac{G_k(x)}{G_k(qx)} = \frac{g_0(qx)}{d_1} \quad ,$$

2. $$\Bigg(\frac{q^2 d_{k+1} g_{k-1}(q^{k+1}x)\ldots g_0(q^{k+1}x)B_0(q^{k+1}x) - q_k(q^{k+2}x)\ldots g_0(q^{k+2}x)B_0(q^{k+}}{(1-q)^2 q^{2k+3}x^2}$$

$$+d_{k+1}a_k - a_{k+1}\Big)\frac{g_k(q^{k+1}x)G_k(x)}{d_{k+1}^2}$$

$$= \left(\frac{q^2 d_1 B_0(qx) - g_0(q^2x)B_0(q^2x)}{(1-q)^2 q^3 x^2} + d_1 a_0 - a_1\right)\frac{g_0(qx)}{d_1^2} \qquad k \in \mathbf{N}$$

The first condition can be reduced to

$$g_k(x) = \frac{d_{k+1}}{d_1} g_0(x) \quad .$$

From this we infer that the equation 2 are equivalent to

$$\left(q^2 d_1 \mathbf{1} - g_0(qx)Q\right)\left(\mathbf{1} - (qd_1)^{-2k} g_0(q^k x)\dots g_0(qx) g_0^k(q^k x) Q^k\right) B_0(x) =$$

$$= (1-q)^2 q x^2 \left(\frac{d_{k+1}a_k - a_{k+1}}{d_{k+1}\dots d_1} d_1^{-k+1} g_0(q^k x)\dots g_0(qx) - d_1 a_0 + a_1\right) \ ,$$

$k \in \mathbb{N}$. We have the following series of equations for the function B_0. We make the assumption that the series reduces to one equation.

This requirement is equivalent to the following

1. $g_0(x) = d_1 q^{\gamma} \qquad \gamma \in \mathbb{R},$
2. $B_0(x) = x^2 \left(b_2 + b_1 x^{-\gamma} + b_0 x^{-2\gamma}\right)\ ,$
3. $a_{k+1} = d_{k+1} \dots d_1 q^{-\gamma k} \left(-a_0 \frac{[\gamma k]}{[\gamma]} + \right.$

$$+ \left(\frac{a_1}{d_1} \frac{[\gamma(k+1)]}{[\gamma]} - q b_2 [\gamma k][\gamma(k+1)\right), \qquad k \in \mathbb{N}$$

Substituting B_0 to (*) we obtain

$$\alpha_0(x) = \frac{q^{\gamma+1} b_2}{(1-q)^2} + \frac{q^{\gamma}(d_1 a_0 - a_1)}{(1-q^{\gamma}) d_1} + h x^{-\gamma} + \frac{q^{1-\gamma} b_0}{(1-q)^2} x^{-2\gamma}\ ,$$

Finally we obtain the solutions

$$
\begin{aligned}
B_k(x) &= q^{k\gamma} d_k \dots d_1 B_0(x) \ , \\
\eta_k(x) &= q^{k\gamma} d_k \dots d_1 \eta_0(q^{-k}x) \ , \\
\varphi_k(x) &= q^{k\gamma} \varphi_0(q^{-k}x) \ , \\
\alpha_k(x) &= q^{-k\gamma} d_k \dots d_1 \alpha_0(q^{-k}x) \ .
\end{aligned}
$$

$$
A_k(x) = q^{k\gamma} d_k \dots d_1 \left(q^{-k} A_0(q^{-k}x) + [-2k] b_2 x + \right.
$$

$$
\left. + [k(\gamma-2)] b_1 x^{-\gamma+1} + [2k(\gamma-1)] b_0 x^{-2\gamma+1} \right) \ ,
$$

$$
f_k(x) = q^{-k\gamma} f_0(q^{-k}x) - \frac{1-q^{k(1-\gamma)}}{(1-q)x} \ ,
$$

$$
\varrho_k(x) = \frac{q^{-\frac{\gamma k(k-1)}{2}}}{d_k d_{k-1}^2 \dots d_1^k} \frac{\varrho_0(q^{-k}x)}{\prod_{n=-k+1}^{0} B_0(q^n x)},
$$

where

$$
\begin{aligned}
f_0(x) &= \sqrt{\frac{\alpha_0(x)}{B_0(x) - (1-q)xA_0(x)}} - \frac{1}{(1-q)x}, \\
\varrho_0(x) &= \frac{B_0(qx)}{B_0(x) - (1-q)xA_0(x)} \varrho_0(qx).
\end{aligned}
$$

The creation and annihilation operators are in this case given by

$$\mathbf{A}_k = \partial_q - \frac{1}{(1-q)x} + q^{-\gamma k}\sqrt{\frac{\alpha_0(q^{-k}x)}{\eta_0(q^{-k}x)}}\ ,$$

$$\mathbf{A}_k^* = d_k \dots d_1 \left(-q^{\gamma k}(b_2 x^2 + b_1 x^{2-\gamma} + b_0 x^{2-2\gamma}) \left(\partial_q Q^{-1} + \frac{1}{(1-q)x} \right) + \right.$$

$$\left. + \sqrt{\alpha_0(q^{-k}x)\eta_0(q^{-k}x)} \right)$$

Hamiltonian takes the form

$$\mathbf{H}_k = d_k \dots d_1 \left(-(1-q)q^{-1}x^3 B_0(x) \sqrt{\frac{\alpha_0(q^{-(k+1)}x)}{\eta_0(q^{-(k+1)}x)}} \partial_q Q^{-1} \partial_q + \right.$$

$$+ \left(-q^{-1}x^2(b_2 + b_1 x^{-\gamma} + b_0 x^{-2\gamma}) \sqrt{\frac{\alpha_0(q^{-(k+1)}x)}{\eta_0(q^{-(k+1)}x)}} + \sqrt{\alpha_0(q^{-k}x)\eta_0(q^{-k}x} \right.$$

$$+ \frac{b_2 + b_1 x^{-\gamma} + b_0 x^{-2\gamma}}{(1-q)^2} \left(q - (1-q)x \sqrt{\frac{\alpha_0(q^{-(k+1)}x)}{\eta_0(q^{-(k+1)}x)}} \right) +$$

$$+ q^{-\gamma k} \alpha_0(q^{-k}x) - \frac{1}{(1-q)x} \sqrt{\eta_0(q^{-k}x)\alpha_0(q^{-k}x)} +$$

$$\left. -q^{-\gamma(k-1)} \left(a_0 \frac{[\gamma(k-1)]}{[\gamma]} - \frac{a_1}{d_1} \frac{[\gamma k]}{[\gamma]} + q b_2 [\gamma(k-1)][\gamma k] \right) \right) ,$$

Example: class of q–Hahn polynomials

$$\boxed{f_k(x) \equiv 0,\ k \in \mathbb{N} \cup \{0\}} \quad \wedge \quad \boxed{d_k = q^{-1}}$$

These conditions can be reduced to

$$B_0(x) = b_2 x^2 + b_1 x + b_0 \ ,$$
$$A_0(x) = \left((1+q)b_2 - q(a_0 - \frac{a_1}{d_1}) \right) x + \frac{b_1}{1-q} - (1-q)h \ .$$

We have $\gamma = 1$,

$$\begin{aligned} B_k(x) &= B_0(x) \ , \\ A_k(x) &= \left(q^{-k} A_0(q^{-k}x) + \frac{1-Q^{-k}}{(1-q)x} B_0(x) \right) = \widetilde{a_k} x + \widetilde{b_k} \ , \end{aligned}$$

where

$$\begin{aligned} \widetilde{a_k} &= -q^{1-2k} \left(q[2(k-1)]b_2 + (a_0 - q a_1) \right) \ , \\ \widetilde{b_k} &= \frac{b_1}{1-q} - (1-q) q^{-k} h \ . \end{aligned}$$

In this case the annihilation and creation operators are given by

$$\mathbf{A}_k = \partial_q \ ,$$
$$\mathbf{A}_k^* = -(b_2 x^2 + b_1 x + b_0)\partial_q Q^{-1} - \widetilde{a_k} x - \widetilde{b_k}$$

and the Hamiltonian

$$H_k = -B_0(x)\partial_q Q^{-1}\partial_q - A_k(x)\partial_q +$$
$$+q^{-2k+1}\left(-a_0[k-1] + q a_1[k] - q b_2[k-1][k]\right) \ .$$

- Let $\varrho_0(x)$ be the solution of the q–Pearson equation $\partial_q\left(B_0\varrho_0\right) = A_0\varrho_0$. Then $\varrho_k(x) = \frac{\varrho_0(q^{-k}x)}{B_0(q^{-k+1}x)\ldots B_0(x)}$ is the solution of q–Pearson equation with functions $A_k(x)$, $B_k(x)$.
- ϱ_0 and ϱ_k satisfy the same boundary conditions.
- If $\{P_k^n(x)\}_{n=0}^{\infty}$ is the orthogonal polynomial system corresponding to the Hamiltonian $\mathbf{H}_k$ then $\{\partial_q P_k^n(x)\}_{n=0}^{\infty}$ is the orthogonal polynomial system corresponding to $\mathbf{H}_{k-1}$.

Considered equation has the form

$$\left(B_0\partial_q Q^{-1} + A_k\right)\partial_q P_k^n = \lambda_k^n P_k^n$$

and we obtain the solutions of the Hahn equation

$$P_k^0 = 1 \ ,$$

$$P_k^n(x) = \mathbf{A}_k^* \mathbf{A}_{k-1}^* \dots \mathbf{A}_{k-n+1}^* \mathbf{1} \quad n = 1, 2, ...k \ ,$$

for eigenvalues

$$\lambda_k^0 = 0$$

$$\lambda_k^n = \widetilde{a}_k[n] + b_2[n][n-1]q^{-(n-1)} \ .$$

Example

$\boxed{\varrho_k(x) = const}$ This is equivalent to these conditions

$$\gamma = 1 \quad \wedge \quad b_2 = b_1 = 0 \, ,$$

$$\gamma = 2 \quad \wedge \quad b_2 = b_0 = 0 \, .$$

Example 2
q–deformation of the harmonic oscillator

$\boxed{\gamma = 1}$

$\boxed{d_k = q^{-1}}$ $\boxed{b_0 = 1}$

$$B_k(x) = 1 \; ,$$

$$A_k(x) = 0 \; ,$$

$$f_k(x) = q^{-k} f_0(q^{-k} x) \; ,$$

$$\varrho_k = \varrho_0 = 1 \; ,$$

where

$$f_0(x) = \sqrt{\frac{q^2(q^{-1}a_0 - a_1)}{(1-q)} + h\frac{1}{x} + \frac{1}{(1-q)^2}\frac{1}{x^2}} - \frac{1}{(1-q)x} \; .$$

In this case the annihilation and creation operators are given by

$$\mathbf{A}_k = \partial_q + q^{-k} f_0(q^{-k}x) \ ,$$

$$\mathbf{A}_k^* = -\partial_q Q^{-1} + q^{-k} f_0(q^{-k}x) \ .$$

and Hamiltonian has the form

$$\mathbf{H}_k = -\left(1 + (1-q) q^{-k-1} x f_0(q^{-k-1}x)\right) \partial_q Q^{-1} \partial_q +$$

$$+q^{-k}\left(f_0(q^{-k}x) - q^{-1} f_0(q^{-k-1}x)\right) \partial_q +$$

$$-q^{-k} \partial_q (f_0(q^{-k-1}x)) + q^{-2k} f_0^2(q^{-k}x) + q^{-2k}\left(a_0 + (q^2 a_1 - a_0)[k]\right) \ .$$

Solutions of the equations

$$\mathbf{A}_k \psi_k^0 = 0$$

have the form

1. $$\psi_k^0(x) = \frac{C_k^0}{\sqrt{\left(\frac{q^{-k}x}{x_1};q\right)_\infty \left(\frac{q^{-k}x}{x_2};q\right)_\infty}} \quad \text{for} \quad q^{-1}a_0 - a_1 \neq 0$$

   where x_1 i x_2 are the roots of the polynomial

   $$(1-q)q^2(q^{-1}a_0 - a_1)x^2 + (1-q)^2 hx + 1 = 0\ .$$

2. $$\psi_k^0(x) = \frac{C_k^0}{\sqrt{\left(-(1-q)^2 \frac{h}{b_0} q^{-k}x;q\right)_\infty}} \quad \text{for} \quad q^{-1}a_0 = a_1 \quad h \neq 0\ .$$

The operators $\mathbf{A}_k$, $\mathbf{A}_k^*$, Q and Q^{-1} satisfy the relations

$$q\mathbf{A}_k^* Q^{-1} = Q^{-1}\mathbf{A}_{k-1}^* \ ,$$
$$\mathbf{A}_k^* Q = qQ\mathbf{A}_{k+1}^* \ ,$$
$$q\mathbf{A}_k Q^{-1} = Q^{-1}\mathbf{A}_{k-1} \ ,$$
$$\mathbf{A}_k Q = qQ\mathbf{A}_{k+1} \ .$$

It easy to see that the operator Q^{-1} acts as follows

$$\psi_0^0 \xrightarrow{\frac{C_1^0}{C_0^0}Q^{-1}} \psi_1^0 \xrightarrow{\frac{C_2^0}{C_1^0}Q^{-1}} \dots \xrightarrow{\frac{C_k^0}{C_{k-1}^0}Q^{-1}} \psi_k^0 \xrightarrow{\frac{C_{k+1}^0}{C_k^0}Q^{-1}} \dots \ .$$

The functions ψ_k^0 are eigenvectors of the Hamiltonians $\mathbf{H}_k$ with the eigenvalues

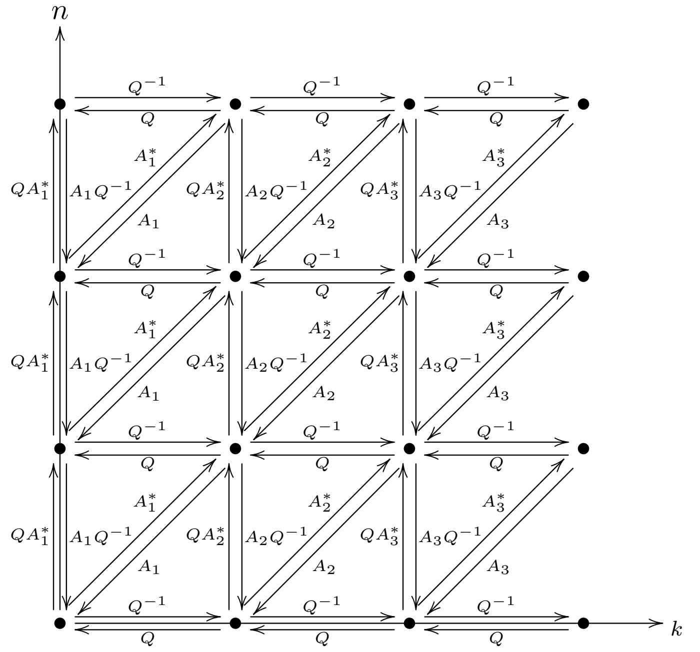
$$\lambda_k^0 = a_k = q^{-2k}\left(a_0 + (q^2 a_1 - a_0)[k]\right) \ .$$

Similarly it is easy to show that the functions

$$\psi_k^n(x) = Q^{-k}\psi_0^n(x)$$

are eigenvectors of $\mathbf{H}_k$ with

$$\lambda_k^n = q^{-2k}\left(\lambda_0^n + (q^2 a_1 - a_0)[k]\right) \ .$$



n
k
Q^{-1}
Q
A_1^*
A_2^*
A_3^*
QA_1^*
QA_2^*
QA_3^*
A_1Q^{-1}
A_2Q^{-1}
A_3Q^{-1}
A_1
A_2
A_3

The functions

$$\psi_k^n(x) = \frac{1}{\sqrt{(a_0 - qa_1)^n n_q! q^{n(n-1)+k}}} Q^{n-k} \mathbf{A}_n^* \dots \mathbf{A}_1^* \psi_0^0(x) \ ,$$

for $k \in \mathbb{N} \cup \{0\}$ and $n \in \mathbb{N} \cup \{0\}$, are the eigenvectors of Hamiltonians corresponding to the eigenvalues

$$\lambda_k^n = q^{-2k+n} \left(a_0 + (q^2 a_1 - a_0)[k-n] \right) \ .$$

It is easy to show that in the limit $q \to 1$ this case gives us the harmonic oscillator

$$\mathbf{H}_k = -\frac{d^2}{dx^2} + \frac{(a_0 - a_1)^2}{4} x^2 + \frac{a_1 + a_0}{2} + (a_1 - a_0)k$$

with eigenvectors

$$\psi_k^n(x) = \left(-\frac{d}{dx} + \frac{a_0 - a_1}{2} x \right)^n e^{-\frac{a_0 - a_1}{4} x^2} \quad \text{for} \quad n \in \mathbb{N} \cup \{0\}$$

corresponding to the eigenvalues

$$\lambda_k^n = a_0 + (a_0 - a_1)(n - k) \ .$$

$$
\begin{aligned}
\mathbf{A}_k =& \frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} \partial_q + f_k, \\
\mathbf{A}_k^* =& - \frac{q^{k(\gamma-1)}}{[\gamma]} x^{\gamma-1} B_k \partial_q Q^{-1} + B_k f_k - \\
& - A_k \left(1 + (1 - q^{\gamma}) q^{k\gamma} x^{\gamma} f_k \right), \\
a_k =& - q^{1-k} \left(a_0 [k-1] - a_1 [k] + q^{\gamma} b_2 \frac{[k-1][k]}{[\gamma]^2} \right),
\end{aligned}
$$

where

$$B_k(x) = q^{2k\gamma-k}x^{2(\gamma-1)}B(q^kx),$$

$$A_k(x) = \frac{q^{k\gamma-k}}{1-q^\gamma}x^{\gamma-2}\left(B(q^kx) - q^{2k(1-\gamma)}B(x)\right),$$

$$f_k(x) = \frac{q^{-k+\frac{\gamma-1}{2}}}{(1-q^\gamma)x^\gamma}\sqrt{\frac{D(qx)}{B(x)}} - \frac{1}{(1-q^\gamma)q^{k\gamma}x^\gamma},$$

and

$$B(x) = b_2x^2 + b_1x + b_0,$$

$$D(x) = \left(b_2 + (1-q^\gamma)[\gamma]q^{-\gamma}(a_0 - a_1)\right)x^2 + \\ (b_1 + (1-q^\gamma)c)\,x + b_0,$$

$$[k] = \frac{1-q^k}{1-q} = 1 + q + \ldots q^{k-1}.$$

Having in mind the factorization property

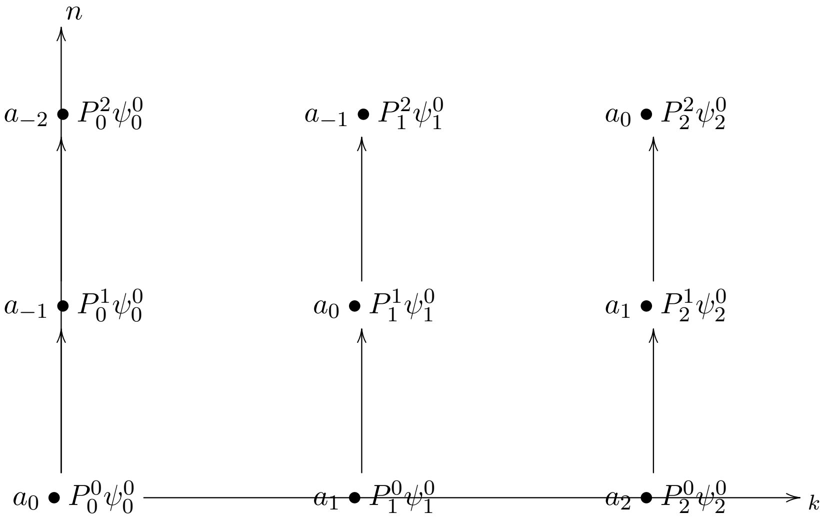
$$\mathbf{A}_k^*\mathbf{A}_k + a_k = Q^{-1}\mathbf{A}_{k+1}\mathbf{A}_{k+1}^* Q + a_k$$

we can look for the solutions of the eigenvalue problem in the form

$$\psi_k^n(x) = P_k^n(x)\psi_k^0(x)$$

under the condition

$$\lambda_k^n = a_{k-n} \ .$$


n
a_{-2} $P_0^2\psi_0^0$
a_{-1} $P_1^2\psi_1^0$
a_0 $P_2^2\psi_2^0$
a_{-1} $P_0^1\psi_0^0$
a_0 $P_1^1\psi_1^0$
a_1 $P_2^1\psi_2^0$
a_0 $P_0^0\psi_0^0$
a_1 $P_1^0\psi_1^0$
a_2 $P_2^0\psi_2^0$
k

We obtain the second order q–difference equation (Hahn equation)

$$-q^{-1}D(qx)P_k^n(qx) - B(q^kx)P_k^n(q^{-1}x) + $$
$$+ \left(q^{-1}D(qx) + B(q^kx)\right) P_k^n(x) =$$
$$= (1-q^n)\left(qb_2 + (1-q^\gamma)[\gamma]q^{-\gamma+1}(a_0-a_1) - \right.$$
$$\left. -q^{2k-n}b_2\right) x^2 P_k^n(x)$$

solutions to which could be expressed in terms of the basics hypergeometric series

$${}_3\Phi_2\left(\begin{matrix} a_1, a_2, a_3 \\ d_1, d_2 \end{matrix}\,\middle|\, q; x\right) = \sum_{k=0}^{\infty} \frac{(a_1;q)_k(a_2;q)_k(a_3;q)_k}{(d_1;q)_k(d_2;q)_k(q;q)_k} x^k$$
$$(a;q)_k = (1-a)(1-qa)\dots(1-q^{k-1}a)$$
$$(a;q)_\infty = (1-a)(1-qa)(1-q^2a)\dots$$

$$\partial_q \left(\frac{q^{-k\gamma}}{[\gamma]} x^{1-\gamma} B_k \varrho_k \right) = A_k \varrho_k$$ q–Pearson equation

$$x^{1-2\gamma} B_k \varrho_k |\psi_k^0|^2 \; |_a^b = 0$$ boundary conditions

$$\begin{aligned} B(x) =& b_2 x^2 + b_1 x + b_0 = b_0 \left(1 - \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right), \\ D(x) =& \left(b_2 + (1-q^\gamma)[\gamma] q^{-\gamma}(a_0 - a_1)\right) x^2 + \\ & (b_1 + (1-q^\gamma)c)\, x + b_0 = \\ & = b_0 \left(1 - \frac{x}{y_1}\right)\left(1 - \frac{x}{y_2}\right). \end{aligned}$$

The case of the Big q–Jacobi orthogonal polynomials

($b_2 \neq 0$, $b_0 \neq 0$ and $b_2 + (1-q^{\gamma})[\gamma]q^{-\gamma}(a_0 - a_1) \neq 0$)
The solutions are

$$\psi_k^0(x) = C x^{(k+\frac{1}{2})(\gamma-1)} \sqrt{\frac{\left(\frac{x}{x_1};q\right)_\infty \left(\frac{x}{x_2};q\right)_\infty}{\left(\frac{qx}{y_1};q\right)_\infty \left(\frac{qx}{y_2};q\right)_\infty}} \, ,$$

$$P_k^n(x) = {}_3\phi_2\left(\begin{array}{c} q^{-n}, q^{-2k+n+1}\frac{x_1 x_2}{y_1}y_2, \frac{q}{y_1}x \\ q^{-k+1}\frac{x_2}{y_1}, q^{-k+1}\frac{x_1}{y_1} \end{array} \middle| q; q \right)$$

and the eigenvalue is

$$\lambda_k^n = a_0[n-k+1] - q a_1 [n-k] + \frac{q^{\gamma} b_2}{[\gamma]^2}[n-k+1][k-n] \, ,$$

the weight function

$$\varrho_k(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(\frac{x}{x_1};q\right)_{k+1}\left(\frac{x}{x_2};q\right)_{k+1}} \ ,$$

q–interval

$$[a,b]_q = [q^{-k}x_1, q^{-k}x_2]_q \ .$$

In the limit $q \to 1$ (we have assumed that b_2 and b_1 do not depend on the parameter q):

$$
{}^1\psi_k^0(x) = C x^{(\gamma-1)(k+\frac{1}{2})} \, (x - x_1)^{\frac{1-\gamma^2(a_0-a_1)}{2} + \frac{\frac{\gamma^2 b_1(a_0-a_1)}{2b_2} - \gamma c}{2\sqrt{b_1^2 - 4b_2 b_0}}} \times
$$

$$
\times \, (x - x_2)^{\frac{1-\gamma^2(a_0-a_1)}{2} - \frac{\frac{\gamma^2 b_1(a_0-a_1)}{2b_2} - \gamma c}{2\sqrt{b_1^2 - 4b_2 b_0}}} \, ,
$$

the Jacobi orthogonal polynomials

$$
{}^1P_k^n(x) = P_n^{(\alpha_k,\beta_k)}(y) = \frac{(\alpha_k+1)_n}{n!} \times
$$

$$
\times {}_2F_1\left(\begin{matrix} -n, n+\alpha_k+\beta_k+1 \\ \alpha_k+1 \end{matrix} \,\middle|\, \frac{1-y}{2} \right) \, ,
$$

where

$$
\alpha_k = -\frac{\gamma^2(a_0-a_1)}{b_2} - k - \frac{\gamma(\gamma(a_0-a_1)x_1 + c)}{b_2(x_2-x_1)} \, ,
$$

$$
\beta_k = -1 + \frac{\gamma(\gamma(a_0-a_1)x_1 + c)}{b_2(x_2-x_1)} \, .
$$

After the transformation given by

$$x = \sqrt{\frac{\Delta}{4b_2^2}} \cosh\left(\gamma^{-1}\sqrt{b_2}(z-c)\right) - \frac{b_1}{2b_2}$$

or

$$x = \sqrt{\frac{\Delta}{4b_2^2}} \sin\gamma^{-1}\sqrt{|b_2|}z - \frac{b_1}{2b_2} \ ,$$

we obtain the Schrödinger equation with the **Rosen–Morse II potential** (for $b_2 > 0$)

$$V_k(z) = D_1 \coth\gamma^{-1}\sqrt{b_2}(z-c)\,\mathrm{cosech}^2\,\gamma^{-1}\sqrt{b_2}(z-c) +$$

$$+ D_2\,\mathrm{cosech}^2\,\gamma^{-1}\sqrt{b_2}(z-c) + D_3$$

or the **Eckart– II potential** (for $b_2 < 0$)

$$V_k(z) = D_1 \tan\gamma^{-1}\sqrt{|b_2|}z\,\mathrm{sech}\,\gamma^{-1}\sqrt{|b_2|}z +$$

$$+ D_2\,\mathrm{sech}^2\,\gamma^{-1}\sqrt{|b_2|}z + D_3 \ ,$$

where D_1 and D_2 depend on the parameters $b_2, b_1, b_0, a_0, a_1, c, \gamma$.

The case of the Big q–Laguerre orthogonal polynomials

$(b_2 \neq 0,\, b_0 \neq 0,\, b_2 + (1-q^{\gamma})[\gamma]q^{-\gamma}(a_0 - a_1) = 0$ and $b_1 + (1-q^{\gamma})c \neq 0)$
The solutions are

$$\psi_k^0(x) = Cx^{(k+\frac{1}{2})(\gamma-1)}\sqrt{\frac{\left(\frac{x}{x_1};q\right)_\infty\left(\frac{x}{x_2};q\right)_\infty}{\left(-\left(\frac{b_1}{b_0}+(1-q^{\gamma})\frac{c}{b_0}\right)qx;q\right)_\infty}}\,,$$

$$P_k^n(x) = {}_3\phi_2\left(\begin{array}{c} q^{-n}, 0, -\left(\frac{b_1}{b_0}+(1-q^{\gamma})\frac{c}{b_0}\right)qx \\ -\left(\frac{b_1}{b_0}+(1-q^{\gamma})\frac{c}{b_0}\right)q^{1-k}x_1, -\left(\frac{b_1}{b_0}+(1-q^{\gamma})\frac{c}{b_0}\right)q^{1-k}x_2 \end{array}\middle| q;q\right)$$

and the eigenvalue is

$$\lambda_k^n = q^{k-n}\left(a_0[n-k+1] - a_1[n-k]\right)\,,$$

the weight function

$$\varrho_k(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(\frac{x}{x_1};q\right)_{k+1}\left(\frac{x}{x_2};q\right)_{k+1}} ,$$

q–interval

$$[a,b]_q = [q^{-k}x_1, q^{-k}x_2]_q ,$$

$$[a,b]_q = [q^{-k}x_1, q^{-k}x_2]_q .$$

In the limit $q \to 1$ (we have assumed that b_1 does not depend on the parameter q):

$$ {}^1\psi_k^0(x) = C x^{\frac{\gamma-1}{2}-k} \left(x + \frac{b_0}{b_1} \right)^{\frac{b_1 - c + \gamma^2 (a_0 - a_1) \frac{b_0}{b_1}}{2b_1}} \times $$
$$ \times \exp\left(-\frac{\gamma^2(a_0 - a_1)}{2b_1} x \right) , $$

the Laguerre polynomials

$$ {}^1P_k^n(x) = L_n^{(\alpha_k)}(y) = \frac{(\alpha_k+1)_n}{n!} {}_1F_1 \left(\begin{matrix} -n \\ \alpha_k + 1 \end{matrix} \middle| y \right) , $$

where

$$ \alpha_k = \frac{\gamma^2(a_0 - a_1) b_0}{b_1^2} - \gamma c + k , $$
$$ \varrho_k^1(x) = \frac{x^{(2k+1)(1-\gamma)}}{(x + \frac{B_0}{b_1})^{k+1}} , $$
$$ [a, b]_1 = [-\frac{b_0}{b_1}, \infty] , $$

After the transformation given by

$$x = \frac{\gamma^{-2} b_1}{4} z^2 - \frac{b_0}{b_1} \ ,$$

we obtain the Schrödinger equation with the **three–dimensional isotropic harmonic oscillator potential**

$$V_k(z) = D_1 z^2 + \frac{D_2}{z^2} + D_3 \ ,$$

where D_1 and D_2 depend on the parameters $b_1, b_0, a_0, a_1, c, \gamma$.

The case of the Al–Salam–Carlitz II orthogonal polynomials

$(b_2 \neq 0,\ b_0 \neq 0,\ b_2 + (1-q^\gamma)[\gamma]q^{-\gamma}(a_0 - a_1) = 0$ and $b_1 + (1-q^\gamma)c = 0)$

The solutions are

$$\psi_k^0(x) = C x^{(k+\frac{1}{2})(\gamma-1)} \sqrt{\left(\frac{x}{x_1};q\right)_\infty \left(\frac{x}{x_2};q\right)_\infty}\ ,$$

$$P_k^n(x) = \left(-\frac{x_1}{x_2}\right)^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{array}{c} q^{-n}, \left(\frac{q^k}{x_2}\right)^{-1} \\ 0 \end{array} \middle| q; \frac{q^{k+1}}{x_1}x\right)$$

and the eigenvalue is

$$\lambda_k = q^{k-n}\left(a_0[n-k+1] - a_1[n-k]\right)\ ,$$

the weight function

$$\varrho_k(x) = \frac{x^{(1-\gamma)(2k+1)}}{\left(\frac{x}{x_1};q\right)_{k+1}\left(\frac{x}{x_2};q\right)_{k+1}}\ ,$$

q–interval

In the limit $q \to 1$

$$^1\psi_k^0(x) = Cx^{\frac{\gamma-1}{2}-k} \exp\left(-\frac{\gamma^2(a_0-a_1)}{b_0}x^2 - \frac{\gamma c}{2b_0}x\right) ,$$

the Hermite polynomials

$$^1P_k^n(x) = H_n(y) = (2y)^n {}_2F_0\left(\begin{array}{c} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{array} \middle| -\frac{1}{y^2}\right) ,$$

$$\begin{aligned} \varrho_k^1(x) =& x^{(2k+1)(1-\gamma)} , \\ [a,b]_1 =& [-\infty, \infty] . \end{aligned}$$

After the transformation given by

$$x = \gamma^{-1}\sqrt{b_0}z$$

we obtain the Schrödinger equation with the **harmonic oscillator potential**

$$V_k(z) = D_1 z^2 + D_2 ,$$

where D_1 and D_2 depend on the parameters b_0, a_0, a_1, c, γ.

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