Donagi-Witten construction and a graded covering of a supermanifold

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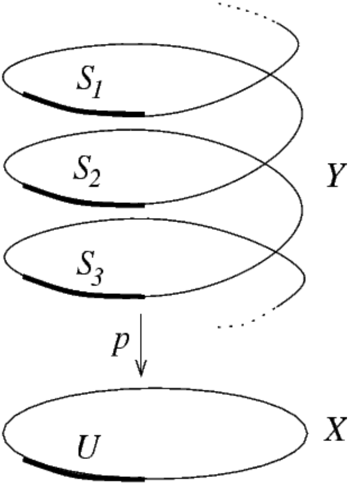
Coverings in geometry

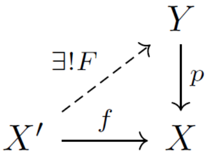


$p : \mathbb{R} \to S^1$, where $p(t) = e^{it}$.

Properties of a universal covering

- For any smooth connected manifold X **there exists a universal covering** $p : Y \to X$, which is unique up to isomorphism.

- Let X' be connected, simply connected. Any smooth map $f : X' \to X$, **factors through** Y:



Cover(ing)s in algebra.

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All modules are over a ring R.

Definition. (**A cover of a module**) If F is a family of modules, then an F-**cover** of a module M is a module C together with a surjective homomorphism

$$p : C \to M$$

with the following properties:

- C is in the family F;
- Any (surjective) homomorphism $f : N \to M$, where $N \in F$, factors through C;
- Any endomorphism of C commuting with the map p is an automorphism. (This implies that C is unique up to isomorphism.)

Example 1. (Torsion-free cover.) Over an integral domain, every module M has a torsion-free cover $C \to M$.

$$C \in F = \{\text{torsion-free modules}\} \subset \{\text{all modules}\}.$$

Example 2. (Flat cover.) Flat cover conjecture (Enochs 1981).

For a general ring, every module M has a flat cover C.

The conjecture resolved positively simultaneously by Bican, El Bashir and Enochs (2001).

Donagi–Witten construction.

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In fact this is a functor from the category of supermanifolds to the category of double vector bundles.

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In fact this is a functor from the category of supermanifolds to the category of double vector bundles.

Related functors were constructed independently in

(1) *Jotz Lean, M.* N-manifolds of degree 2 and metric double vector bundles, arXiv:1504.00880.
(2) *del Carpio-Marek. F.* Geometric structure on degree 2 manifolds. PhD-thesis, IMPA, Rio de Janeiro, 2015.

(3) *A. Bruce, J. Grabowski and M. Rotkiewicz*, Polarisation of Graded Bundles. SIGMA 12 (2016), 106, 30 pages.

(4) *E. Vishnyakova.* Graded manifolds of type Δ and n-fold vector bundles, Letters in Mathematical Physics 109 (2), 2019, 243-293.

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Denote the Donagi-Witten functor by $\mathbf{F}_2$. Using ideas of papers (3)-(4) we can generalize the Donagi-Witten construction and obtain the following functors $\mathbf{F}_n$ for any n:

$$\{\textbf{supermanifolds}\} \xrightarrow{\mathbf{F}_n} \{\textbf{graded manifolds of degree n}\}$$

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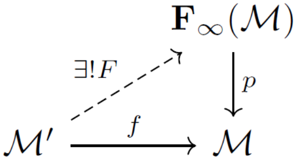
Our contribution: We realized that a limit functor $\mathbf{F}_\infty$ is a $\mathbb{Z}^{\geq 0}$-graded covering of a supermanifold $\mathcal{M}$.

Our contribution:

We defined a $\mathbb{Z}^{\geq 0}$**-graded cover(ing)** for any supermanifold, which has properties of geometrical and algebraic cover(ing):

- Let $\mathcal{M}'$ be a graded manifold. Any morphism $f : \mathcal{M}' \to \mathcal{M}$, where $\mathcal{M}$ is a supermanifold, **factors through** $\mathbf{F}_\infty(\mathcal{M})$:



- It is unique up to isomorphism.
- In some sense the cover map is a "local diffeomorphism".

Example of a covering, Lie superalgebra $\mathfrak{gl}_{m|n}(\mathbb{K})$.

$$\mathfrak{gl}_{m|n}(\mathbb{K}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\},$$

$\mathbf{F}_\infty(\mathfrak{gl}_{m|n}(\mathbb{K}))$ contains all matrices in the following form

$$\begin{pmatrix} A_1 & 0 & 0 & 0 & \cdots \\ C_1 & D_1 & 0 & 0 & \cdots \\ A_2 & B_1 & A_1 & 0 & \cdots \\ C_2 & D_2 & C_1 & D_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

Here A_i and D_j are copies of A and D, respectively. The same for B_s and C_t.

Such or related algebras were studied under the name "loop algebras" or "covering superalgebra" for instance in

1. *Nicoletta Cantarini*, $\mathbb{Z}$-graded Lie superalgebras of infinite depth and finite growth, Annali della Scuola Normale Superiore di Pisa- Classe di Scienze 1 (3):545-568 (2002).
2. *Bruce Allison, Stephen Berman, John Faulkner, and Arturo Pianzola*, Realization of graded-simple algebras as loop algebras, Forum Math. 20 (2008), no. 3, 395–432.
3. *Alberto Elduque, Graded-simple algebras and cocycle twisted loop algebras.* Proc. Amer. Math. Soc. 147 (2019), 2821-2833.

Example. ($\mathbb{Z}^{\geq 0}$-graded covering of $\mathbb{R}^{0|1}$.)

By definition, $\mathbb{R}^{0|1} = (pt, \bigwedge(\xi))$.

In this case $\mathbf{F}_\infty(\mathbb{R}^{0|1}) := (pt, \bigwedge(\eta_1, \eta_3, \ldots))$ is a $\mathbb{Z}^{\geq 0}$-graded covering of $\mathbb{R}^{0|1}$ with the covering morphism

$$p = (p_0, p^*) : \mathbf{F}_\infty(\mathbb{R}^{0|1}) \to \mathbb{R}^{0|1},$$

where

$$p^*(F) = F(pt) + \frac{\partial F}{\partial \xi}\eta_1 + \frac{\partial F}{\partial \xi}\eta_3 + \cdots, \quad F \in \bigwedge(\xi).$$

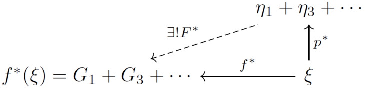
This formula implies that

$$p^*(const) = const, \quad p^*(\xi) = \eta_1 + \eta_3 + \cdots .$$

$\mathbf{F}_\infty(\mathbb{R}^{0|1})$ is a $\mathbb{Z}^{\geq 0}$-graded manifold with $deg(\eta_{2k+1}) = 2k+1$.

Universal properties of $\mathbf{F}_\infty(\mathbb{R}^{0|1})$.

- If $\mathcal{M}'$ is a $\mathbb{Z}^{\geq 0}$-graded manifold and $f : \mathcal{M}' \to \mathbb{R}^{0|1}$ is a morphism of supermanifolds (i.e. f is $\mathbb{Z}_2^{\geq 0}$-graded), then f factors through $\mathbf{F}_\infty(\mathbb{R}^{0|1})$ and the lift of f is unique:



- "local diffeomorphism" $\xi \mapsto \eta_i$, where $i = 1, 3, \ldots$.
- $\mathbf{F}_\infty(\mathbb{R}^{0|1})$ is unique up to isomorphism.

Definition (E.V.)

A $\mathbb{Z}^{\geq 0}$**-covering of a supermanifold** $\mathcal{M}$ is a $\mathbb{Z}^{\geq 0}$-graded manifold $\mathcal{P}$ of infinite degree with $\mathcal{P}_0 = \mathcal{M}_0$ together with a morphism

$$p : \mathcal{P} \to \mathcal{M}$$

such that we can choose atlases $\{\mathcal{U}_i\}$ and $\{\mathcal{V}_i\}$ on $\mathcal{M}$ and $\mathcal{P}$, respectively, with the same base space $(\mathcal{U}_i)_0 = (\mathcal{V}_i)_0$, with even and odd coordinates (x_a, ξ_b) in $\mathcal{U}_i$ and with graded coordinates (y_a^s, η_b^t), where s is an even integer and t is an odd integer, in $\mathcal{V}_i$ such that

$$pr_s \circ p^*(x_a) = y_a^s, \quad pr_t \circ p^*(\xi_b) = \eta_b^t,$$

where $pr_k : \mathcal{O}_\mathcal{P} \to (\mathcal{O}_\mathcal{P})_k$ is the natural projection.

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Theorem (E.V.)

For any supermanifold $\mathcal{M}$ *the graded manifold* $\mathbf{F}_\infty(\mathcal{M})$ *is a* $\mathbb{Z}^{\geq 0}$*-covering of* $\mathcal{M}$*.*