On integrability, geometrisation and knots

Alexander P. Veselov

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To dear memory of Anatol Odzijewicz

First School in Geometry and Physics, Bialowieza, 2012

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XXXI WGMP, Bialowieza, 2012

Plan

- Thurston's geometries
- Liouville-Arnold integrability revisited
- Chaos and integrability in $SL(2,\mathbb{R})$-geometry
- Geodesics on the modular 3-fold and knot theory

References

V.I. Arnold *Some remarks on flow of line elements and frames*. Sov. Math. Dokl. **138:2** (1961), 255-257.

W.P. Thurston *Hyperbolic geometry and 3-manifolds.* In: LMS Lecture Notes Series **48**, CUP, 1982.

É. Ghys *Knots and Dynamics.* Intern. Congress of Math. Vol. 1. Eur. Math. Soc., Zurich 2007, 247-277.

A. Bolsinov, A.Veselov, Y. Ye *Chaos and integrability in* $SL(2,\mathbb{R})$*-geometry.* Russian Math. Surveys **76(4)** (2021), 557-586.

Felix Klein (1849-1925) and Henri Poincaré (1854-1912)

Every conformal class of surface metrics has complete constant curvature representative.

In particular, for any Riemannian genus $g > 1$ surface is conformally equivalent to a quotient $\mathbb{H}^2/G$ by a discrete subgroup $G \subset PSL(2, \mathbb{R})$.

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Explicit example: sphere with $n = 3$ punctures, $G = \Gamma_2 \subset PSL(2, \mathbb{Z})$ (corollary: **Picard's theorem**).

Dimension 3: Thurston's geometrization programme

 

William Thurston (1946-2012) and Grigori Perelman (1966-)

Every closed 3-manifold can be decomposed into pieces such that each admits one of the following eight types of geometric structures of finite volume

$$\mathbb{E}^3,\ \mathbb{S}^3,\ \mathbb{S}^2\times\mathbb{R},\ \mathbb{H}^2\times\mathbb{R},\ Nil,\ Sol,\ \widetilde{SL(2,\mathbb{R})},\ \mathbb{H}^3.$$

$$Nil = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} \quad Sol = \left\{ \begin{pmatrix} e^x & 0 & y \\ 0 & e^{-x} & z \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

and $\widetilde{SL(2,\mathbb{R})}$ is the universal cover of $SL(2,\mathbb{R})$.

Bianchi 1898: Classification of 3D real Lie algebras: types $I - IX$. Among them the unimodular Lie algebras are of types

- I: abelian $\mathbb{R}^3$
- II: nilpotent (Heisenberg) *nil*
- VI_0: solvable 2D Poincare $sol \cong e(1,1)$
- VII_0: solvable 2D Euclidean $e(2)$
- $VIII$: simple $sl(2,\mathbb{R})$
- IX: simple $so(3)$.

Thurston 1970s: Classification of special geometries in 3D:

Euclidean E^3*, spherical* S^3*, hyperbolic* $\mathbb{H}^3$*, the product type* $S^2 \times \mathbb{R}$ *and* $\mathbb{H}^2 \times \mathbb{R}$ *and three geometries related to 3D Lie groups* Nil, Sol, $SL(2,\mathbb{R})$

Special geometry here means local homegeneity (any two points have isometric neighbourhoods), existence of compact models and some maximality condition.

Bianchi versus Thurston

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What's about integrability of the corresponding geodesic flows?

Arnold 1963: Hamiltonian system on symplectic manifold M^{2n} is integrable in Liouville sense if it has n independent integrals $F_1, \dots, F_n$ in involution.

When the joint integral level

$$M_c = \{x \in M^{2n} : F_i(x) = c_i,\ i = 1, \dots, n\}$$

is non-critical and compact, then it must be a torus T^n with quasi-periodic dynamics and in its vicinity one can introduce "action-angle" variables I_i, φ_i with $H = H(I)$: $\dot{I} = 0, \dot{\varphi} = \omega(I)$.

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Tomei 1984, Gaifullin 2006: Natural compactification of the integral level in the (extended) Toda system is aspherical manifold, which can be used as the universal in Steenrod's cycle realisation problem!

Sol-case: chaotic critical level

In *Sol*-case the principal examples are mapping tori M_A^3 of the hyperbolic maps $A : T^2 \to T^2$, $A \in SL(2, \mathbb{Z})$ (first considered by Poincaré in 1892!):

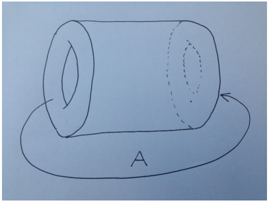

Figure: Torus mapping of A

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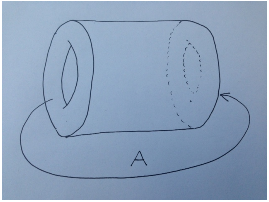

Figure: Torus mapping of A

Bolsinov and Taimanov 2000: On *Sol*-manifolds M^3_A the geodesic flow is Liouville integrable in smooth category, but not in analytic one.

At the degenerate level the system is chaotic (Anosov map), so the system has positive topological entropy!

$SL(2,\mathbb{R})$-case

In $SL(2,\mathbb{R})$-case the principal examples are unit tangent bundles of hyperbolic surfaces

$$\mathcal{M}^3_\Gamma = \Gamma\backslash PSL(2,\mathbb{R}) = S\mathcal{M}^2_\Gamma, \quad \mathcal{M}^2_\Gamma = \Gamma\backslash\mathbb{H}^2,$$

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Bolsinov, Veselov and Ye 2021: The corresponding phase space $T^*\mathcal{M}^3_\Gamma$ contains two open regions with integrable and chaotic behaviour.

In the integrable region we have Liouville integrability with analytic integrals, while in the chaotic region the system is not Liouville integrable even in smooth category and has positive topological entropy.

Naturally reductive metrics: left $SL(2,\mathbb{R})$- and right $SO(2)$-invariant

$$\langle X, Y\rangle = \alpha(\operatorname{sym} X, \operatorname{sym} Y) + \beta(\operatorname{skew} X, \operatorname{skew} Y), \ \ \alpha > 0 > \beta,$$

$$(X, Y) := \operatorname{Tr} XY, \ \operatorname{skew} X := (X - X^\top)/2 \in so(n), \ \operatorname{sym} X := (X + X^\top)/2.$$

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Setting $\alpha = 2$, we have the inner product with

$$|\Omega|^2 = 4(u^2 + vw) + k(v - w)^2, \quad k = 1 - \frac{\beta}{\alpha} > 1$$

on the Lie algebra

$$\Omega = \begin{pmatrix} u & v \\ w & -u \end{pmatrix} \in sl(2,\mathbb{R}).$$

$PSL(2,\mathbb{R})$ can be identified with the unit tangent bundle $S\mathbb{H}^2$ of the hyperbolic plane $\mathbb{H}^2 = SL(2,\mathbb{R})/SO(2)$:

$$g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2,\mathbb{R}) \longrightarrow (z = \frac{ai+b}{ci+d}, \xi = \frac{i}{(ci+d)^2}) \in S\mathbb{H}^2,$$

where $\mathbb{H}^2$ is realised as the upper half-plane $z = x + iy,\ y > 0$ with the hyperbolic metric $ds^2 = dzd\bar{z}/y^2$.

$PSL(2,\mathbb{R})$ as unit tangent bundle of hyperbolic plane

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In coordinates $x, y, \varphi = \arg \xi$ the metric has the form

$$ds^2 = \frac{dx^2 + dy^2}{y^2} + (k-1)(d\varphi + \frac{dx}{y})^2,$$

which is the generalised **Sasaki metric** on $S\mathbb{H}^2$, considered by **Nagy 1977**. Sasaki metric corresponds to $k = 2$ and can be considered as the "best one".

The general Euler-Poincare equations of the corresponding geodesic flow have

$$\dot{M} = [M, \Omega],$$

where $\Omega := g^{-1}\dot{g} \in \mathfrak{g}$ and $M \in \mathfrak{g}^* \cong \mathfrak{g}$ is determined by $(\Omega, M) = \langle \Omega, \Omega \rangle$.

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In our case we have $2M = (\alpha + \beta)\Omega + (\alpha - \beta)\Omega^\top$, so the Euler-Poincare equations have the form

$$\dot{M} = \frac{\beta - \alpha}{2\alpha\beta}[M, M^\top],$$

which can be easily integrated explicitly (e.g. **Mielke 2002**).

Euler-Poincare equations and geodesics on $SL(2,\mathbb{R})$

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The geodesics on $SL(2,\mathbb{R})$ with $\Omega(0) = \Omega_0$ can be explicitly given by

$$g(t) = g(0)e^{tX_0}e^{tY_0},$$

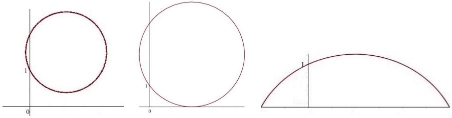
where

$$X = \frac{1}{\alpha}M = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad Y = \frac{\alpha-\beta}{2\beta}\begin{pmatrix} 0 & b-c \\ c-b & 0 \end{pmatrix}.$$

Nagy 1977, BVY 2021: The projection of the geodesics on $PSL(2,\mathbb{R}) = S\mathbb{H}^2$ to $\mathbb{H}^2$ are curves of constant geodesic curvature

$$\kappa = \frac{b-c}{\sqrt{4a^2+(b+c)^2}}.$$

They are circles if $\kappa^2 > 1$, or arcs of circles if $\kappa^2 \le 1$ and can be described as magnetic geodesics on $\mathbb{H}^2$ in constant magnetic field with density $B = b - c$:



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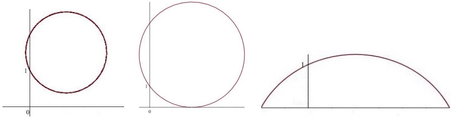

Magnetic geodesics on $\mathcal{M}^2_\Gamma = \mathbb{H}^2/\Gamma$: **Caratheodory 1932, Hedlund 1936, Arnold 1961, Paternain 1997, Taimanov 2004.**
Arnold 1961: the entropy is $h = \sqrt{1-\kappa^2}$ if $\kappa^2 \leq 1$ (and 0 otherwise).

Liouville integrability for Fuchsian quotients M^3_Γ

Let $\Gamma \subset PSL(2,\mathbb{R})$ be a Fuchsian group such that $\Gamma\backslash\mathbb{H}^2 = \mathcal{M}^2_\Gamma$ has finite area and consider the quotient $\mathcal{M}^3_\Gamma = \Gamma\backslash PSL(2,\mathbb{R}) = S\mathcal{M}^2_\Gamma$.

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We have two obvious integrals of geodesic flow on $G = SL(2,\mathbb{R})$: the Hamiltonian $H = \frac{1}{2}(\Omega, M) = \frac{\alpha}{4\beta}(\beta[4a^2 + (b+c)^2] - \alpha(b-c)^2)$ and the Casimir function $\Delta = \det M = a^2 + bc$. As the third integral one can take any Γ-invariant function of the right momentum $m = gMg^{-1}$.

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However, it is known that corresponding Γ-action is discrete if $\Delta = \delta < 0$ and has some dense orbits if $\Delta = \delta > 0$ (**Hedlund 1936, Dal'Bo 2011**).

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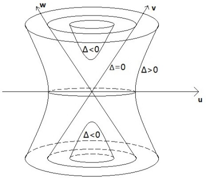
Corollary: The geodesic flow on $T^*\mathcal{M}^3_\Gamma$ has no smooth right-invariant integrals F independent from Δ in the part of the phase space $T^*\mathcal{M}^3_\Gamma$ with $\Delta \geq 0$.

In the domain $\Delta < 0$ we can use any real analytic automorphic function as the additional third analytic integral F.

Interpret $\mathbb{R}^3$ as the set of binary quadratic forms $Q(x,y) = Ax^2 + 2Bxy + Cy^2$, then the degeneracy condition $D := B^2 - AC = 0$ defines a conic in the corresponding projective plane.
The disc with $D < 0$ is the Cayley-Klein projective model of the hyperbolic plane, while the points with $D > 0$ by polarity correspond to the lines in this model.



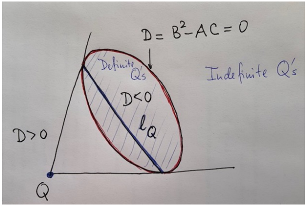


Figure: $sl(2,\mathbb{R})$-symplectic leaves and Klein's correspondence

Consider now the special case of modular group $\Gamma = SL(2,\mathbb{Z})$ and its principal congruence subgroup Γ_2.

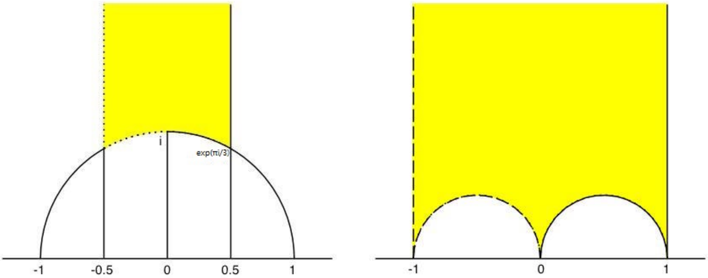


Figure: The fundamental domains of Γ and Γ_2

Let $\Gamma = SL(2,\mathbb{Z})$ be the modular group and consider the *modular 3-fold*

$$\mathcal{M}^3 = SL(2,\mathbb{R})/SL(2,\mathbb{Z}).$$

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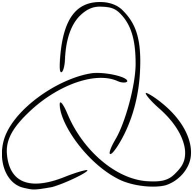
There is a remarkable observation due to **Quillen (1970s):**

$$\mathcal{M}^3 = SL(2,\mathbb{R})/SL(2,\mathbb{Z}) = S^3 \backslash \mathcal{K},$$

where $\mathcal{K}$ is the trefoil knot:



Milnor, 1972: Note first that $\mathcal{M}^3 = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ can be interpreted as the moduli space of the elliptic curves $\mathbb{C}/\mathcal{L}$ up to real scaling. The corresponding $\wp$-function satisfies the Weierstrass equation

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3,$$

which defines an elliptic curve if and only if the discriminant

$$D = g_2^3 - 27g_3^2 \neq 0.$$

The intersection of the unit sphere $S^3 \subset \mathbb{C}^2(g_2, g_3)$ with the set $D = 0$ is $(2,3)$-torus ($=$ trefoil) knot.

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Alternatively, the projection $\mathcal{M}^3 \to \mathcal{M}^2 = \mathbb{H}^2/PSL(2,\mathbb{Z})$ is the Seifert fibration with two singular fibres corresponding to orbifold points of order 2 and 3 of $\mathcal{M}^2$. The missing Hopf fibre over infinity is thus (2,3)-torus knot.

E. Artin, 1924: *Periodic geodesics on modular surface $\mathcal{M}^2$ are labelled by integer binary quadratic forms Q.*
Their lifts to $\mathcal{M}^3 = S\mathcal{M}^2$ form certain knots called by Ghys *modular*.

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Birman and Williams (1983), Ghys (2006): Modular knots are exactly those, which appear as periodic orbits in the celebrated Lorenz system

$$\begin{cases} \dot{x} = \sigma(-x+y) \\ \dot{y} = rx - y - xz \\ \dot{z} = -bz + xy \end{cases}, \quad \sigma = 10,\ b = 8/3,\ r = 28.$$

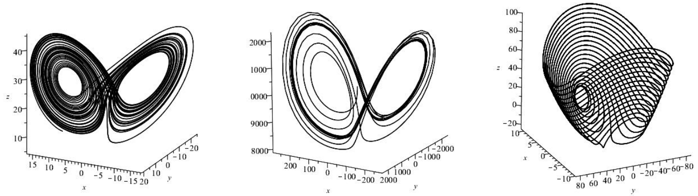

Figure: The Lorenz trajectories for $r = 28$, 10000 and $r = \infty$

Consider the integral

$$\mathcal{C} := \kappa^2 = \frac{(b-c)^2}{4a^2+(b+c)^2} = \frac{\beta H - \alpha\beta\Delta}{\beta H - \alpha^2\Delta}$$

of the geodesic flow on $\mathcal{M}^3$. We have seen that the system is integrable if $\mathcal{C} > 1$ and non-integrable otherwise.

When $\mathcal{C} = 0$ we have the lifts of the geodesics on the modular surface $\mathcal{M}^2$ considered by Ghys.

It is natural to ask what happens when $\mathcal{C} > 1$.

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BVY 2021: The periodic geodesics on modular 3-fold $\mathcal{M}^3_\Gamma$ with sufficiently large values of $\mathcal{C}$ represent the trefoil cable knots in $S^3 \setminus \mathcal{K}$.
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Integrable limit and cable knots

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Note that the Lorenz system for large r has at most 3 periodic orbits, each representing a trivial knot (**Moiseev, Neishtadt 1995**).

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Geometric classification of knots

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Torus knot $K_{p,q}$ specified by a pair of coprime integers p and q lies on the surface of a solid torus in $\mathbb{R}^3$, winding p times around the axis of rotation of the torus and q times around the central circle of the torus. Trefoil knot $\mathcal{K} = K_{2,3}$.

Satellite knots can be get in the following way: let K_1 be a knot inside an unknotted solid torus and knot the torus in the shape of another knot K_2. In the special case of K_1 being a torus knot, we have *cable knots of* K_2.

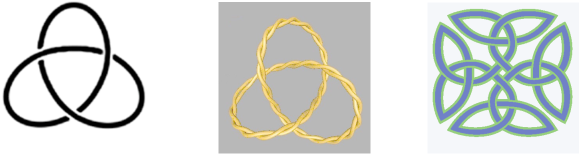


Figure: Trefoil knot $\mathcal{K}$, its $(2, 33)$ cable knot and celtic satellite knot

Thurston (1978): Every knot is either torus, or a satellite, or hyperbolic knot.

Torus knot $K_{p,q}$ specified by a pair of coprime integers p and q lies on the surface of a solid torus in $\mathbb{R}^3$, winding p times around the axis of rotation of the torus and q times around the central circle of the torus. Trefoil knot $\mathcal{K} = K_{2,3}$.

Satellite knots can be get in the following way: let K_1 be a knot inside an unknotted solid torus and knot the torus in the shape of another knot K_2. In the special case of K_1 being a torus knot, we have *cable knots of* K_2.

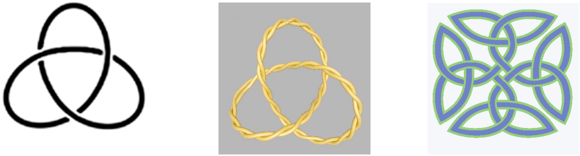


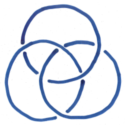
Figure: Trefoil knot $\mathcal{K}$, its $(2, 33)$ cable knot and celtic satellite knot

Complements to the torus knots admit $SL(2, \mathbb{R})$-structure, hyperbolic knots - $\mathbb{H}^3$-structure, while the satellite knots do not admit any geometric structure.

Let $\Gamma_2 \subset SL(2, \mathbb{Z})$ consist of matrices congruent to the identity modulo 2:

$$\mathcal{M}_2^3 = \Gamma_2 \backslash SL(2, \mathbb{R}) \cong S^3 \backslash \mathcal{L},$$

where $\mathcal{L}$ is the **Hopf 3-link**



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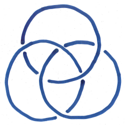

In the integrable domain with large $\mathcal{C}$, when the ratio of frequencies is rational, we have the invariant torus filled by the torus knots $K_{p,q}$.

What can we say about knots in $\mathcal{M}^3 = S^3 \setminus \mathcal{K}$ appearing at other levels of $\mathcal{C}$?
Can any prime knot be realised this way?

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Not all modular knots are hyperbolic (although many of them are), but their links with the trefoil knot $\mathcal{K}$ are always hyperbolic: **Foulon, Hasselblat 2013**.

Volume Conjecture **(Kashaev 1997, Murakami et al 2002)**: for hyperbolic knots

$$Vol(S^3 \setminus K) = 2\pi \lim_{N\to\infty} \frac{\ln |J_N(K)|}{N},$$

where $J_N(K)$ is the *Jones polynomial* of K evaluated at $e^{2\pi i/N}$.

Some results about the corresponding volumes for modular knots: **Brandts, Pinsky, Silberman 2019**.

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Note also that iterated torus knots are precisely the knots with zero topological entropy: **Llibre and MacKay 1990**.

They also have interesting relations with the theory of double affine Hecke algebras: **Berest and Samuelson, Cherednik and Danilenko 2016**.

Thurston's wisdom