Banach Poisson-Lie groups

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Outline

Reference :

- A.B.Tumpach, *Banach Poisson-Lie groups and Bruhat-Poisson structure of the restricted Grassmannian*, Communications in Mathematical Physics, 2020.
- D. Beltiţă, T. Goliński, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics, 2018.

Finite-dimensional Poisson–Lie groups

References for finite-dimensional Poisson-Lie groups

V.G. Drinfel'd, '83
Y. Kosmann-Schwarzbach, F. Magri, '88
J.-H. Lu, '91

Poisson–Lie groups in the finite-dimensional case

Manin triples

$\Updownarrow$

Lie-bialgebras

$\Updownarrow$

connected simply connected Poisson–Lie groups

Poisson–Lie groups

Let us start with an example of a Manin triple...

$\mathfrak{u}(n)$ = Lie-algebra of the unitary group $\mathrm{U}(n)$
= space of skew-symmetric matrices

$\mathfrak{b}(n)$ = Lie-algebra of the Borel group $\mathrm{B}(n, \mathbb{C})$
= space of upper triangular matrices with real coef. on diagonal

Then the space $M(n, \mathbb{C}) = \mathfrak{gl}(n, \mathbb{C})$ of all complex matrices decomposes :

$$M(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$$

and for the non-degenerate symmetric bilinear continuous map $\langle \cdot, \cdot \rangle$ given by

$$\langle A, B \rangle = \mathrm{Im}\,\mathrm{Tr}(AB) = \text{ imaginary part of trace}(AB)$$

the blocks $\mathfrak{u}(n)$ and $\mathfrak{b}(n)$ are both isotropic.

Poisson–Lie groups

Definition of a Manin triple

A **Banach Manin** triple consists of a triple of Banach Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ over a field $\mathbb{K}$ and a **non-degenerate symmetric bilinear** continuous map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ on $\mathfrak{g}$ such that

1. the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$ is invariant with respect to the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ of $\mathfrak{g}$, i.e.

$$\langle [x, y]_{\mathfrak{g}}, z \rangle_{\mathfrak{g}} + \langle y, [x, z]_{\mathfrak{g}} \rangle_{\mathfrak{g}} = 0, \quad \forall x, y, z \in \mathfrak{g}; \tag{1}$$

2. $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as Banach spaces;
3. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Banach Lie subalgebras of $\mathfrak{g}$;
4. both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the bilinear map $\langle \cdot, \cdot \rangle_{\mathfrak{g}}$.

Let M be a finite-dimensional manifold.

Poisson bracket

A **Poisson bracket** on M is a bilinear map
$\{\cdot,\cdot\} : \mathscr{C}^\infty(M) \times \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M)$ with

- skew-symmetry $\{f,g\} = -\{g,f\}$
- Jacobi identity $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
- Leibniz rule $\{f,gh\} = \{f,g\}h + g\{f,h\}$

Poisson tensor

$\{f,g\} = \pi(df,dg)$ where $\pi \in \Gamma(\Lambda^2 TM)$ is a bivector field

Example

Any symplectic manifold is a Poisson manifold

Let G be a finite-dimensional Lie group.

Poisson-Lie groups

A **Poisson-Lie group** G is a Lie group equipped with a Poisson structure compatible with the group multiplication.

Example

Any Lie group G with $\{\cdot, \cdot\} = 0$ is a Poisson Lie group
Any compact Lie group, like $SU(n)$, is a Poisson-Lie group in a non-trivial way.

Definition

For any Poisson-Lie group (G, π), with Lie algebra $\mathfrak{g}$, one defines $\Pi_r^G := R_g^* \pi \ : G \to \Lambda^2 \mathfrak{g}$ as

$$\Pi_r^G(g)(\alpha, \beta) := \pi(R_g^* \alpha, R_g^* \beta), \quad \alpha, \beta \in \mathfrak{g}^*$$

Let (G, π) be a finite-dimensional Poisson-Lie group.

Facts

1. the fact that the Poisson tensor π is compatible with the group multiplication implies the following cocycle condition on $\Pi_r^G := R_g^* \pi$
$$\Pi_r^G(gh) = \Pi_r^G(g) + \mathrm{Ad}_g \Pi_r^G(h)$$
2. the derivative of $\Pi_r^G \ : G \to \Lambda^2\mathfrak{g}$ at the unit of the group is a cocycle $\theta \ : \mathfrak{g} \to \Lambda^2\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$
$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = \; & \theta(y)(\mathrm{ad}_x^*\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}_x^*\beta) \\ & -\theta(x)(\mathrm{ad}_y^*\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}_y^*\beta)\end{aligned}$$
where $x, y \in \mathfrak{g}$ and $\alpha, \beta \in \mathfrak{g}^*$.
3. the Jacobi identity verified by the Poisson structure implies that $\theta^* := [\cdot,\cdot]_{\mathfrak{g}}^* \ : \Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$

Finite-dimensional Lie bialgebras

Lie bialgebra

Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. One says that $(\mathfrak{g}, \mathfrak{g}^*)$ form a **Lie bialgebra** if there is a Lie bracket $\Lambda^2\mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ whose dual map $\mathfrak{g} \to \Lambda^2\mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\Lambda^2\mathfrak{g}$

$$\begin{aligned}\theta\left([x,y]\right)(\alpha,\beta) = \; & \theta(y)(\mathrm{ad}^*_x\alpha,\beta) + \theta(y)(\alpha,\mathrm{ad}^*_x\beta) \\ & -\theta(x)(\mathrm{ad}^*_y\alpha,\beta) - \theta(x)(\alpha,\mathrm{ad}^*_y\beta)\end{aligned}$$

Poisson–Lie groups in the finite-dimensional case

Manin triples

$\Updownarrow$

Lie-bialgebras

$\Updownarrow$

connected simply connected Poisson–Lie groups

Example

- $M(n,\mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{b}(n)$ with $\langle A, B\rangle = \mathrm{Im}\ \mathrm{Tr} AB$ is a Manin triple.
- U(n) and B($n,\mathbb{C}$) are dual Poisson-Lie groups with

$$\Pi_r^G(g)(x_1, x_2) = \Im \mathrm{Tr}\, p_{\mathfrak{u}}(g^{-1}x_1 g) p_{\mathfrak{b}}(g^{-1}x_2 g).$$

- Moreover $GL(n,\mathbb{C}) = U(n) \times B(n)$ because of Iwasawa dec.
- This gives a dressing action

$$\varphi : B(n) \times U(n) \to U(n)$$

by $\varphi(b)(k) = k'$ where k' is the unique element of $U(n)$ such that $bk = k'b'$ with $b' \in B(n)$.

Reference :

J.-H. Lu, A. Weinstein, *Poisson Lie groups, Dressing Transformations, and Bruhat Decompositions*, Journal of Differential Geometry, 1990.

Poisson manifold modelled on a non-separable Banach space

Problems :

(1) no bump functions available (norm not even $\mathscr{C}^1$ away from the origin)

(2) there exist derivations of order greater then 1 [Kriegl, Michor, '97]

(3) there exist Poisson bracket without Poisson tensor (Leibniz rule does not imply existence of Poisson tensor) [Beltita, Golinski, T., 2018]

(4) existence of Hamiltonian vector field is not automatic

References for Poisson geometry on Banach manifods

A. A. Odzijewicz, T. Ratiu, 2003
P. Cabau, F. Pelletier, 2011
K.H.Neeb, H. Sahlmann, T. Thiemann, 2013
de Bièvre, F.Genoud, S. Rota Nodari, 2015
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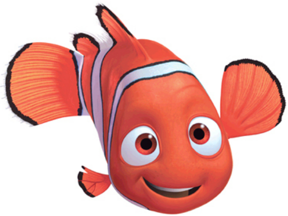
Poisson bracket not given by a Poisson tensor

Queer Poisson Bracket = Poisson bracket not given by a Poisson tensor



Reference :

D. Beltiță, T. Goliński, A.B.Tumpach, *Queer Poisson Brackets*, Journal of Geometry and Physics, 2018.

What are the traps of infinite-dimensional geometry?

In infinite-dimensional geometry, the golden rule is :
"Never believe anything you have not proved yourself!"

- The distance function associated to a Riemannian metric may by the zero function (Example by Michor and Mumford).
- Levi-Cevita connection may not exist for weak Riemannian metrics
- Hopf-Rinow Theorem does not hold in general : geodesic completeness $\neq$ metric completeness
- Darboux Theorem does not apply to weak symplectic forms
- A formally integrable complex structure does not imply the existence of a holomorphic atlas
- the tangent space differs from the space of derivations (even on a Hilbert space)
- a Poisson bracket may not be given by a bivector field (even on a Hilbert space)
- there are Lie algebras that can not be enlarged to Lie groups (Examples by Milnor or Neeb)

Definition of a Banach Poisson manifold

Definition of a Poisson tensor :
M Banach manifold, $\mathbb{F}$ a subbundle of T^*M in duality with TM.
π smooth section of $\Lambda^2\mathbb{F}^*(\mathbb{F})$ is called a Poisson tensor on M with respect to $\mathbb{F}$ if :

1. for any closed local sections α, β of $\mathbb{F}$, the differential $d\left(\pi(\alpha,\beta)\right)$ is a local section of $\mathbb{F}$;
2. (Jacobi) for any closed local sections α, β, γ of $\mathbb{F}$,

$$\pi\left(\alpha, d\left(\pi(\beta,\gamma)\right)\right) + \pi\left(\beta, d\left(\pi(\gamma,\alpha)\right)\right) + \pi\left(\gamma, d\left(\pi(\alpha,\beta)\right)\right) = 0.$$

Definition of a Poisson Manifold :
A Banach Poisson manifold is a triple $(M, \mathbb{F}, \pi)$ consisting of a smooth Banach manifold M, a subbundle $\mathbb{F}$ of the cotangent bundle T^*M in duality with TM, and a Poisson tensor π on M with respect to $\mathbb{F}$.

Banach symplectic manifold

Any Banach symplectic manifold (M, ω) is naturally a generalized Banach Poisson manifold $(M, \mathbb{F}, \pi)$ with

1. $\mathbb{F} = \omega^\sharp(TM)$;
2. $\pi : \omega^\sharp(TM) \times \omega^\sharp(TM) \to \mathbb{R}$ defined by $(\alpha, \beta) \mapsto \omega(X_\alpha, X_\beta)$ where X_α and X_β are uniquely defined by $\alpha = \omega(X_\alpha, \cdot)$ and $\beta = \omega(X_\beta, \cdot)$.

Definition

Consider a duality pairing $\langle \cdot, \cdot \rangle_{\mathfrak{g}_+, \mathfrak{g}_-} : \mathfrak{g}_+ \times \mathfrak{g}_- \to \mathbb{K}$ between 2 Banach. $\mathfrak{g}_+$ is a **Banach Lie–Poisson space with respect to $\mathfrak{g}_-$** if

- $\mathfrak{g}_-$ is a Banach Lie algebra $(\mathfrak{g}_-, [\cdot, \cdot]_{\mathfrak{g}_-})$
- $\mathfrak{g}_-$ acts continuously on $\mathfrak{g}_+ \hookrightarrow \mathfrak{g}_-^*$ by coadjoint action, i.e.

$$\mathrm{ad}^*_\alpha x \in \mathfrak{g}_+,$$

for all $x \in \mathfrak{g}_+$ and $\alpha \in \mathfrak{g}_-$, and $\mathrm{ad}^* : \mathfrak{g}_- \times \mathfrak{g}_+ \to \mathfrak{g}_+$ is continuous.

Banach Poisson-Lie groups

A **Banach Poisson-Lie group** B is a Banach Lie group equipped with a Banach Poisson manifold structure compatible with the multiplication

Proposition

Let B be a Banach Lie group and $(B, \mathbb{B}, \pi)$ a Banach Poisson structure on B. Then B is a Banach Poisson-Lie group if and only if

1. $\mathbb{B}$ is invariant under left and right multiplications by elements in B,
2. the subspace $\mathfrak{u} := \mathbb{B}_e \subset \mathfrak{b}^*$, where e is the unit element of B, is invariant under the coadjoint action of B on $\mathfrak{b}^*$ and the map

$$\begin{array}{rccl} \Pi_r^B : & B & \to & \Lambda^2\mathfrak{u}^* \\ & g & \mapsto & R_{g^{-1}}^{**}\pi_g, \end{array}$$

   is a 1-cocycle on B with respect to the coadjoint representation of B in $\Lambda^2\mathfrak{u}^*$.

Theorem [T] :

Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group. Then $\mathfrak{g}_+$ is a Banach Lie bialgebra with respect to $\mathfrak{g}_-$. The Lie bracket in $\mathfrak{g}_-$ is given by

$$[\alpha_1, \beta_1]_{\mathfrak{g}_-} := T_e\Pi_r(\cdot)(\alpha_1, \beta_1) \in \mathfrak{g}_- \subset \mathfrak{g}_+^*, \quad \alpha_1, \beta_1 \in \mathfrak{g}_- \subset \mathfrak{g}_+^*, \tag{2}$$

where $\Pi_r := R^{**}_{g^{-1}}\pi : G_+ \to \Lambda^2\mathfrak{g}_-^*$, and $T_e\Pi_r : \mathfrak{g}_+ \to \Lambda^2\mathfrak{g}_-^*$ denotes the differential of Π_r at the unit element $e \in G_+$.

Theorem [T] :

Let $(G_+, \mathbb{F}, \pi)$ be a Banach Poisson–Lie group.If the map $\pi^\sharp : \mathbb{F} \to \mathbb{F}^*$ defined by $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in $TG_+ \subset \mathbb{F}^*$, then $\mathfrak{g}_+$ is a Banach Lie–Poisson space with respect to $\mathfrak{g}_- := \mathbb{F}_e$.

Poisson–Lie groups in the infinite-dimensional case

Manin triple

$\Updownarrow$

Banach Lie-bialgebra + Banach Lie-Poisson space

$\Uparrow$

Banach Poisson–Lie group G + $\pi^\sharp(\alpha) := \pi(\alpha, \cdot)$ takes values in TG

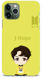
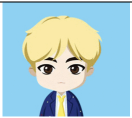
$\mathscr{H}$ separable infinite-dimensional Hilbert space.
On bounded operators $A \in L(\mathscr{H})$ acting on $\mathscr{H}$, define

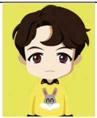
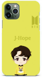
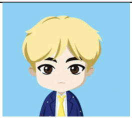
$$\|A\|_p = \left(\mathrm{Tr}(A^*A)^{\frac{p}{2}}\right)^{\frac{1}{p}}$$

For $1 < p < 2 < q < +\infty$, one has:

$$L^1(\mathscr{H}) \subset L^p(\mathscr{H}) \subset L^2(\mathscr{H}) \subset L^q(\mathscr{H}) \subset L(\mathscr{H})$$

For a decomposition, $\mathscr{H} = \mathscr{H}_+ \oplus \mathscr{H}_-$,
$\mathrm{U}_{res}(\mathscr{H}) = \{\left\{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right\} \in U(H), B \text{ and } C \text{ are Hilbert-Schmidt}\}$
$\mathrm{U}_{1,2}(\mathscr{H}) = \{\left\{\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right\} \in U(H), A \text{ and } D \text{ Trace-class}, B \text{ and } C \in L^2(\mathscr{H})\}$
$\mathrm{L}_{1,2}(\mathscr{H}) := \{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), A \text{ and } C \text{ Trace class}, B \text{ and } C \text{ Hilbert-Schmidt}\}$

				
Double Lie group	Group	Lie algebra $\mathfrak{g}$	Fiber $\mathbb{F}_e \subset \mathfrak{g}^*$	Dual Group
$GL(n, \mathbb{C})$	$U(n)$	$\mathfrak{u}(n)$	$\mathfrak{b}(n)$	$B(n)$
$GL_2(\mathscr{H})$	$U_2(\mathscr{H})$	$\mathfrak{u}_2(\mathscr{H})$	$\mathfrak{b}_2(\mathscr{H})$	$B_2(\mathscr{H})$
$GL_p(\mathscr{H}), 1 < p \leq 2$	$U_p(\mathscr{H})$	$\mathfrak{u}_p(\mathscr{H})$	$\mathfrak{b}_p(\mathscr{H})$	$B_p(\mathscr{H})$

				
Double Lie group	Group	Lie algebra $\mathfrak{g}$	Fiber $\mathbb{F}_e \subset \mathfrak{g}^*$	Dual Group
???	$U(\mathscr{H})$	$\mathfrak{u}(\mathscr{H})$	$L^1(\mathscr{H})/\mathfrak{u}_1$	???
??? ???	$U_{res}(\mathscr{H})$ $U_1(\mathscr{H})$	$\mathfrak{u}_{res}(\mathscr{H})$ $\mathfrak{u}_1(\mathscr{H})$	$L^{1,2}(\mathscr{H})/\mathfrak{u}_{1,2}$ $L^1(\mathscr{H})/\mathfrak{u}_1$	??? ???

Example of bounded operator with unbounded triangular truncation [Davidson, Nest Algebras]

$$\begin{pmatrix}
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\ddots & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \frac{1}{n} & \ddots \\
\ddots & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \frac{1}{n-1} & \ddots \\
\ddots & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & & \ddots \\
\ddots & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \frac{1}{3} & \ddots \\
\ddots & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \frac{1}{2} & \ddots \\
\ddots & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & 1 & \ddots \\
\ddots & -\frac{1}{n} & -\frac{1}{n-1} & & -\frac{1}{3} & -\frac{1}{2} & -1 & 0 & \ddots \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}$$

- the triangular truncation is unbounded on the Banach space of trace class operators
- Does there exists a trace class operator whose triangular truncation is not trace class?

Theorem [A.B.T] :

Consider the Banach Lie group $\mathrm{U}_{\mathrm{res}}(H)$, and

1. $\mathfrak{g}_+ := \mathrm{L}_{1,2}(H)/\mathfrak{u}_{1,2}(H) \subset \mathfrak{u}^*_{\mathrm{res}}(H)$,
2. $\mathbb{U} \subset T^*\,\mathrm{U}_{\mathrm{res}}(H)$, $\mathbb{U}_g = R^*_{g^{-1}}\mathfrak{g}_+$,
3. $\tilde{\pi}_r \ : \mathrm{U}_{\mathrm{res}}(H) \to \Lambda^2\mathfrak{g}_+^*$ defined by

$$\tilde{\pi}_r(g)([x_1]_{\mathfrak{u}_{1,2}}, [x_2]_{\mathfrak{u}_{1,2}}) = \Im \mathrm{Tr}(g^{-1}\, p_{\mathfrak{b}_2^+}(x_1)\, g) \left[p_{\mathfrak{u}_2}(g^{-1}\, p_{\mathfrak{b}_2^+}(x_2)\, g)\right],$$

4. $\tilde{\pi}(g) = R_g^{**}\tilde{\pi}_r(g)$.

Then $(\mathrm{U}_{\mathbf{res}}(H), \mathbb{U}, \tilde{\pi})$ **is a Banach Poisson-Lie group.**

Poisson bracket not given by a Poisson tensor

$\mathscr{H}$ separable Hilbert space

Kinetic tangent vector $X \in T_x\mathscr{H}$ equivalence classes of curves $c(t)$, $c(0) = x$, where $c_1 \sim c_2$ if they have the same derivative at 0 in a chart.

Operational tangent vector $x \in \mathscr{H}$ is a linear map $D : C_x^\infty(\mathscr{H}) \to \mathbb{R}$ satisfying Leibniz rule :

$$D(fg)(x) = Df\ g(x) + f(x)\ Dg$$

Poisson bracket not given by a Poisson tensor

Ingredients :

- Riesz Theorem
- Hahn-Banach Theorem
- compact operators $\mathscr{K}(\mathscr{H}) \subsetneq \mathscr{B}(\mathscr{H})$ bounded operators
  $\Rightarrow \exists \ell \in \mathscr{B}(\mathscr{H})^*$ such that $\ell(\mathrm{id}) = 1$ and $\ell_{|\ \mathscr{K}(\mathscr{H})} = 0$.

Queer tangent vector [Kriegl-Michor]

Define $D_x : C^\infty_x(\mathscr{H}) \to \mathbb{R}$, $D_x(f) = \ell(d^2(f)(x))$, where the bilinear map $d^2(f)(x)$ is identified with an operator $A \in \mathscr{B}(\mathscr{H})$ by Riesz Theorem

$$d^2(f)(x)(X, Y) = \langle X, AY \rangle$$

Then D_x is an operational tangent vector at $x \in \mathscr{H}$ of order 2

Poisson bracket not given by a Poisson tensor

Queer tangent vector [Kriegl-Michor]

$$d(fg)(x) = df(x).g(x) + f(x).dg(x)$$

$$\begin{aligned} d^2(fg)(x) &= d^2f(x).g(x) + df(x) \otimes dg(x) \\ &\quad + dg(x) \otimes df(x)) + f(x)d^2g(x) \end{aligned}$$

$$\begin{aligned} D_x(fg) &= \ell(d^2(fg)(x)) \\ &= \ell(d^2f(x)).g(x) + f(x)\ell(d^2g(x)) \\ &\quad + \ell(df(x) \otimes dg(x)) + \ell(dg(x) \otimes df(x)) \\ &= D_xf\ g(x) + f(x)\ D_xg \end{aligned}$$

Poisson bracket not given by a Poisson tensor

Theorem (D. Beltita, T. Golinski, A.B.Tumpach)

Consider $\mathscr{M} = \mathscr{H} \times \mathbb{R}$. *Denote points of* $\mathscr{M}$ *as* (x, λ). *Then* $\{\cdot, \cdot\}$ *defined by*

$$\{f, g\}(x, \lambda) := D_x\left(f(\cdot, \lambda)\right) \frac{\partial g}{\partial \lambda}(x, \lambda) - \frac{\partial f}{\partial \lambda}(x, \lambda) D_x\left(g(\cdot, \lambda)\right)$$

a queer Poisson bracket on $\mathscr{H} \times \mathbb{R}$, *in particular it can not be represented by a bivector field* $\Pi : T^*\mathscr{M} \times T^*\mathscr{M} \to \mathbb{R}$. *The Hamiltonian vector field associated to* $h(x, \lambda) = -\lambda$ *is the queer operational vector field*

$$X_h = \{h, \cdot\} = D_x$$

acting on $f \in C^\infty_x(\mathscr{H})$ *by* $D_x(f) = \ell(d^2(f)(x))$.

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