Talk 3:

5. Higher order analogues of Lie algebroids - comorphism approach to Lie algebroids

① Non-linear algebroids (Voronov)

Th. Voronov, Q-manifolds and Higher Analogs of Lie Algebroids (2010)

Idea: Replace VB $E \to M$ with a $\mathbb{Z}_{\geq 0}$-graded supermanifold:

Def. A non-linear or higher Lie algebroid with base M is non-negatively graded manifold $F \to M$, $M = F_0$, endowed with a homological vector field $Q \in \mathfrak{X}_1(F)$. $Q = Q^a_i\, y^i\, \partial_{x^a} + \underbrace{Q^k(x,y,z)}_{2}\, \underset{-1}{\partial_{y^k}} + \dots$

② HAs - comorphism approach (M. Jóźwikowski, M.R.)

M. Jóźwikowski, M. Rotkiewicz, Prototypes of higher algebroids with applications to variational calculus (2013)

M.J., M.R., Models for higher algebroids (2015)

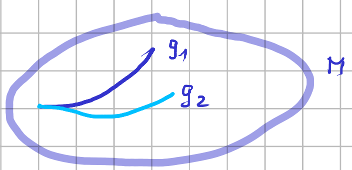
M.J., M.R., Higher-Order Analogs of Lie Algebroids via Vector Bundle Comorphisms (2018)

M.R. Abstract higher order algebroids (2022, not yet published)

Problem:
$$\begin{tikzcd}
\boxed{TM} \arrow[r, dash] \arrow[d, dash] & \boxed{T^k M} \arrow[d, dash] \\
\boxed{\text{Lie algebroids}} \arrow[r, dash] & \boxed{\text{HAs } ?}
\end{tikzcd}$$

Ex. $T^k_e G$, G - Lie group.

$T^k M$ {k^{th} order tangent bundle} consists of k^{th}-order tangency classes of curves in the manifold M.



Non-negatively graded manifold = graded bundles = homogeneity structures

non-negatively graded = $\mathbb{Z}_{\geq 0}$-graded, i.e.

supermanifolds obtained by means of gluing coordinate domains

$(\mathbb{Z}_{\geq 0} \times \mathbb{Z}_2$ - graded$)$
$\qquad\qquad\underset{\text{sign}}{\Downarrow}$

Exmpl. $T^k M$

$M: (x^a) \Rightarrow$ adopted coordinates $(x^a, \dot{x}^a, \ddot{x}^a, \ldots, x^{(a),k})$ on $T^k M$

$$\Big(\underbrace{x^a \circ g(t)}_{= x^a(t)} = x^a(0) + \sum_{j=1}^{n} x^{a,(j)}\, t^j / j! + o(t^k) \Big)$$

$$T^2M: \begin{cases} \underline{x}^a = T^a(x) & (M) \\ \underline{\dot{x}}^a = \dfrac{\partial T^a}{\partial x^b} \cdot \dot{x}^b & \\ \underline{\ddot{x}}^a = \dfrac{\partial x^a}{\partial x^b} \cdot \ddot{x}^b + \dfrac{\partial^2 x^a}{\partial x^b \partial x^c} \cdot \dot{x}^b \cdot \dot{x}^c & \end{cases}$$

weight $\quad 2 \qquad 2 \qquad 1+1$

Homogeneity structure on a manifold M is a monoid action

$$h: \mathbb{R} \times M \to M,$$

(monoid $(\mathbb{R}, \cdot)$ action $\Rightarrow h_{st} = h_s \circ h_t$, where $h_t = h(t, \cdot)$)

Thm (J.G., M.R.) There is one-to-one correspondence

$$\boxed{\text{graded bundles}} \overset{1-1}{\longleftrightarrow} \boxed{\text{homogeneity structures}}$$

From graded bundles to hom. str.:

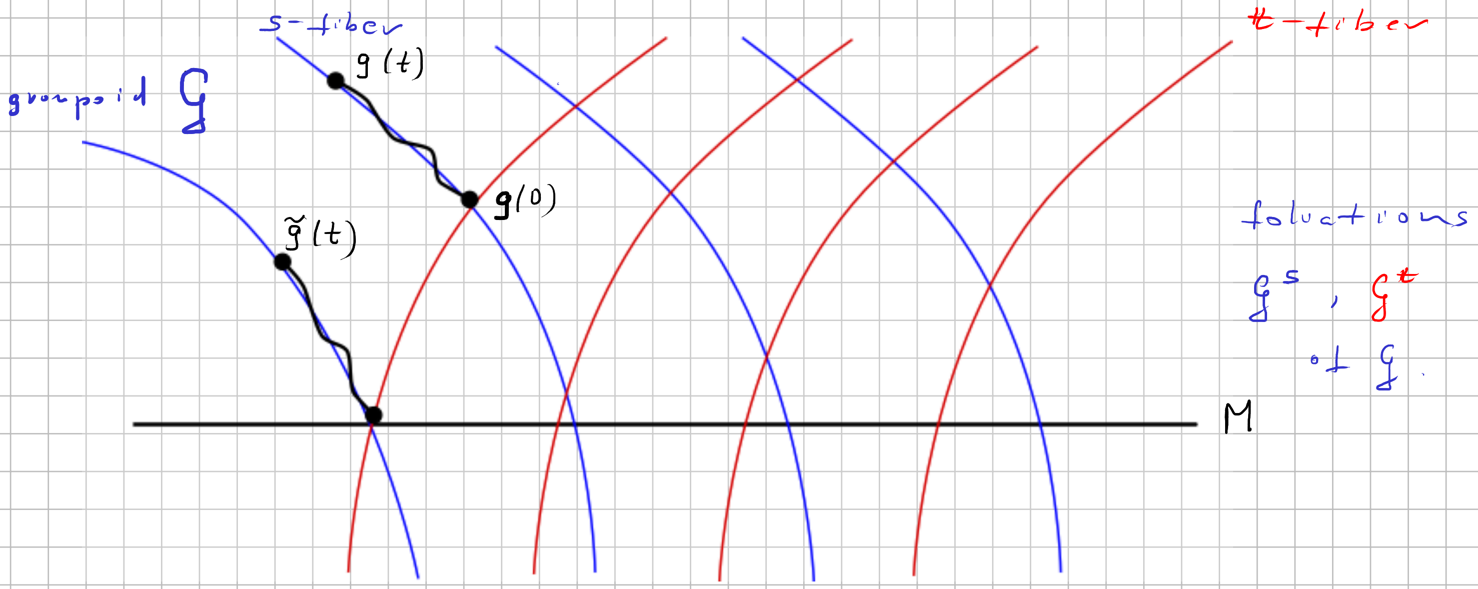
$M: (x^a_w) \Rightarrow \quad h_t: (x^a_w) \longmapsto (t^w x^a_w)$

Back, from HS to GrB: $\quad (F, h) \xrightarrow{\varphi} (T^k F, h_{T^k F})$

$$p \longmapsto [t \mapsto h(t,p)]_k \in T^k F$$

For sufficiently large k, the map φ is embedding.

Reductions of Lie groupoids – a source of natural examples



• $A^k(\mathcal{G}) = T^k_M \mathcal{G}^s$,

••
$$\begin{tikzcd} T^k\mathcal{G}^s \arrow[r, "\mathcal{R}^k"] \arrow[d] & A^k(\mathcal{G}) \arrow[d] \\ \mathcal{G} \arrow[r, "t"] & M \end{tikzcd}$$

••• anchor map $\#^k : A^k(\mathcal{G}) \to T^k M$, $\#^k = (T^k t)\big|_{A^k(\mathcal{G})}$

However, $\Gamma_M(T^k M)$ admits no bracket operation for $k>1$, (no linearity) it is unclear what replaces the bracket of $A^1(\mathcal{G})$.

Solution is obtained by reformulation of the definition of Lie algebroid.

An algebroid as a pair (A, κ)

$(A \to M, [\,,\,], \#)$ Lie algebroid

$\Rightarrow$ $\Lambda = \Lambda_A \in \Gamma(\Lambda^2 T A^*)$ Poisson tensor on A^*

$$\begin{tikzcd} T^*A^* \arrow[r, "\tilde{\Lambda}"] & TA^* \\ T^*A \arrow[u, "I_A", "\simeq"'] \arrow[ur, dashed, "\varepsilon"] & \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} T^*A \arrow[r, "\varepsilon"] \arrow[d] & TA^* \arrow[d] \\ A \arrow[r, "\#"] & TM \end{tikzcd}$$

ε is VB-morph.

$\Rightarrow$ COMORPHISM (Zakrzewski morphism) $\kappa = \varepsilon^*$

In general, a comorphism $r : E_1 \to E_2$, where $\sigma_i : E_i \to M_i$ are fibrations (bundles), $i = 1, 2$, is a relation $r \subset E_1 \times E_2$ (we also write $r : \sigma_1 \to \sigma_2$) of special kind:

$$\begin{tikzcd} E_1 \arrow[r, "r"] \arrow[d, "\sigma_1"'] & E_2 \arrow[d] \\ M_1 & M_2 \arrow[l, "\underline{r}"'] \end{tikzcd}$$

(1) $\underline{r} = r \cap (M_1 \times M_2)$ is a function $M_2 \to M_1$

(2) $r = \bigcup_{y \in M_2} \text{graph}(r_y)$, $\forall_y \; r_y : (E_1)_x \to (E_2)_y$ where $x = \underline{r}(y)$

$\Rightarrow$ A comorphism $r : E_1 \to E_2$ induces a map on the space of sections
$$\hat{r} : \Gamma(E_1) \longrightarrow \Gamma(E_2)$$

Example. $E_i \to M_i$ VBs, $i = 1, 2$, $\varphi : E_1 \to E_2$ is VB-morph.

$$\Rightarrow \quad r = \varphi^* \qquad \begin{tikzcd} E_1^* \arrow[d] & E_2^* \arrow[l] \arrow[d] \\ M_1 \arrow[r] & M_2 \end{tikzcd}$$

This is the case of $\kappa = \varepsilon^*$:

Exr. Show that
(1) If $\mathfrak{g}$ – Lie algebra, then $(X_x, Y_y) \in \kappa$, $(X_x \in T_x\mathfrak{g} \simeq \mathfrak{g})$
$\Leftrightarrow$ $X_x - Y_y = [x, y]$
(2) $A = TM \Rightarrow \kappa = \kappa_M : TTM \circlearrowleft$

$$\begin{tikzcd} TA \arrow[r, "\kappa"] \arrow[d] & TA \arrow[d] \\ A \arrow[r, "\#"] & TM \end{tikzcd}$$
$$(x^a, y^i) \qquad\qquad (\underline{x}^a, \underline{\dot{x}}^a)$$

$$\kappa : \begin{cases} \underline{x}^a = x^a, \quad \underline{\dot{x}}^a = Q^a_i(x)\, \dot{y}^i & (\#) \\ \underline{\dot{y}}^k = y^k + Q^k_{ij}(x) \cdot \underline{y}^i \dot{y}^j \end{cases}$$

Note, κ keeps all the information about algebroid structure on A.

HA structure on $A^k(G)$

The same κ can be obtained in a more natural way

– by reduction of canonical diffeomorphism $\kappa : TTG \to TTG$.

This construction can be easily generalized for HAs:
$$\kappa_{k,G} : T^kTG^s \to TT^kG^s$$

$$\begin{tikzcd} T^kTG^s \arrow[r, "\kappa_{k,G}"] \arrow[d, "T^kR^1"'] & TT^kG^s \arrow[d, "TR^k"] \\ T^kA(G) \arrow[r, dashed, "\kappa_k"] & TA^k(G) \end{tikzcd}$$

($\alpha : [0, \alpha] \times [0, \beta]$ represents elements of T^kT^eM
$$[\alpha]_{k,e} = \frac{d^k}{dt^k}\bigg|_{t=0} \frac{d^e}{ds^e}\, \alpha(t, s) \Big)$$

$\kappa_k \overset{df}{:=}$ image of $\kappa_{k,G}$ under the projection $T^kR^1 \times TR^k$

Thm (Properties of $\kappa_k : T^k A(\mathcal{G}) \to T A^k$)

(1) κ_k is a comorphism,

(2) κ_k is double homogeneous,

(3) κ_k is a Lie subalgebroid of the product of algebroids:

$T^k A(\mathcal{G}) \to T^k M$ (k^{th}-order tangent of $A(\mathcal{G})$)

$T A^k(\mathcal{G})$ (tangent algebroid)

HIGHER ALGEBROIDS — definition

It is natural to consider first more general classes of algebroids

- general algebroids: $\#_L$, $\#_R$ anchors, $[s_1, f s_2] = f[s_1, s_2] + \#_L(s_1)(f)\, s_2$

  $[f s_1, s_2] = f[s_1, s_2] - \#_R(s_2)(f)\cdot s_1$.

- almost Lie algebroids (AL algebroids): (1) Leibniz rule

  (2) $\# : \Gamma(A) \to \mathfrak{X}(M)$ respects $[\,,]_A$ & $[\,,]_{TM}$.

  (compatibility with the bracket of vector fields)

Def. A general algebroid structure on a VB $A \xrightarrow{\sigma} M$ is a linear comorphism $\kappa : T\sigma \to \tau_A$ s.t.

κ induces the identity on the core bundles.

If in addition

(1) $\kappa = \kappa^T \Rightarrow$ skew algebroids ($\underbrace{\#_L = \#_R}_{}$ & $[\,,]$ is skew-symmetric)

are given by base maps of κ, κ^T, resp.

(2) $(T\# \times T\#) : \kappa \Rightarrow \kappa_M$ is a morphism of comorphisms

then κ is AL:

$$\begin{tikzcd} TA \arrow[r, "\kappa"] \arrow[d, "T\#"'] & TA \arrow[d, "T\#"] \\ TTM \arrow[r, "\kappa_M"] & TTM \end{tikzcd}$$

(3) (A, κ) is Lie

if κ is subalgebroid of $T\sigma \times \tau_A$.

DEF Higher algebroids: $(A^k \xrightarrow{\sigma^k} M,\ \kappa : T^k\sigma_1 \to \tau_{A^k})$

$$\begin{tikzcd} T^k A^1 \arrow[r, "\kappa^k"] \arrow[d, "T^k\tau^1"'] & T A^k \arrow[d, "\tau_{A^k}"] \\ T^k M & A^k \arrow[l, "\#^k"'] \end{tikzcd}$$

(1) κ^1 defines a skew algebroid str. on A

(2) (compatibility with κ^k_M) AL

$$\begin{tikzcd} T^k A^1 \arrow[r, "\kappa^k"] \arrow[d, "T^k\#^1"'] & T A^k \arrow[d, "T\#^k"] \\ T^k TM \arrow[r, "\kappa^k_M"] & T T^k M \end{tikzcd}$$

(3) (Lie) κ^k is subalgebroid ($\Leftrightarrow \varepsilon_k$ is a Poisson map)

$\Leftrightarrow \hat{\kappa}^k : \Gamma_{\le 0}(T^k A) \to \mathfrak{X}_{\le 0}(A^k)$ is a graded Lie algebra morph.

* Variational calculus on (Lie) HAs.

* Examples