

Talk 2 (1) de Rham differential and supergeometry (Varieties theorem)

$$M, \Omega(M) = \bigoplus_{k \geq 0} \Omega^k(M), \text{ de Rham differential } d: \Omega(M) \rightarrow \Omega(M)$$

has a nice interpretation: d is a vector field on a supermanifold $\pi TM, d \in \mathfrak{X}(\pi TM)$.

(1) Supermanifolds

Def. A supermanifold $\mathcal{M}, \dim \mathcal{M} = (p|q)$ is a \mathbb{Z}_2 -graded ringed space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ locally isomorphic to a superdomain in $\mathbb{R}^{p|q}$.

Annotations: $\mathcal{O}_{\mathcal{M}}$ is the top. space, $\mathbb{R}^{p|q}$ is the space of super-commutative algebras.

• superdomain = $(U, C_{p|q}^\infty)$,

where $U \subseteq \mathbb{R}^p$ (open subset), $C_{p|q}^\infty(V) = C^\infty(V) \otimes \wedge^{\bullet} (\mathbb{R}^q, \dots, \mathbb{R}^q)$

Annotations: $C^\infty(V)$ are smooth functions on V ; $\wedge^{\bullet} (\mathbb{R}^q, \dots, \mathbb{R}^q)$ is Grassmann algebra $(\xi^1 \xi^1 = \xi^2 \xi^2 = 0)$.

Assume: elt. of $C^\infty(V)$ are even, ξ^i - odd.

Informally, one can think of a supermanifold as a "manifold" with both commuting & anti-commuting coordinates.

Examples A VB $E \rightarrow M$ gives a supermanifold πE

$$E: (x^a, y^i), \begin{cases} \bar{x}^a = T^a(x) \\ \bar{y}^i = y^j T_j^i(x) \end{cases} \Rightarrow \pi E: (x^a, \xi^i), \begin{cases} \bar{x}^a = T^a(x) \\ \bar{\xi}^i = \xi^j T_j^i(x) \end{cases}$$

Annotations: \bar{y}^i is odd, $\bar{\xi}^i$ is odd.

Formally, $\pi E = (M, \mathcal{O}_{\pi E}), \mathcal{O}_{\pi E}(U) = \Gamma_U(\wedge^{\bullet} E^*)$

$C^\infty(U) [\xi^1, \dots, \xi^q]$

Claim. $\pi TM = (M, \Omega_M)$

k de Rham differential is a derivation of Ω_M ,
 $\Rightarrow d \in \mathfrak{X}(\pi M)$.

Example. $M \cong \mathbb{C}P^1, \dim M = (1|2)$

$$(x, \xi_1, \xi_2) \mapsto \left(y = \frac{1}{x} - \frac{1}{x^2} \xi_1 \xi_2, \eta_j = x^{-2} \xi_j \right), j=1,2$$

Weinstein: Lie algebroids as generalized tangent bundle.

⇒ „What can be done with the tangent bundle can be done for algebroids, as well.“

• Algebroid = $(\pi E, Q)$, $Q \in \mathcal{X}_q(\pi E)$.

$\Omega^*(A) = \Gamma(\wedge A^*)$, $d_A: \Omega(A) \hookrightarrow$ characterized by

$$\left\{ \begin{array}{l} (d_A f)(s) = \#(s)(f), \quad f \in C^\infty(\pi) \text{ (0-form)} \\ s \in \Gamma(\mathcal{L}) \\ (d_A \alpha)(s_1, s_2) = (\#s_1)(\alpha(s_2)) - (\#s_2)(\alpha(s_1)) - \alpha([s_1, s_2]_A) \end{array} \right.$$

⇒ d_A determines the Lie algebroid structure, and

d_A is a derivation of $\Omega(A) \Rightarrow d_A \in \mathcal{X}_1(\pi A)$

$$\begin{array}{l} A: (x^a, y^i) \\ \pi A: (x^a, \xi^i) \end{array} \Rightarrow d_A = \xi^i Q_{ij}^a(x) \frac{\partial}{\partial x^a} - \frac{1}{2} Q_{ij}^k \xi^i \xi^j \frac{\partial}{\partial \xi^k}$$

$$[e_i, e_j] = Q_{ij}^k e_k \quad (\text{Bod: } \Gamma(\mathcal{L}) \simeq \mathcal{X}_{-1}(\pi E),$$

$$\#e_i = Q_i^a \frac{\partial}{\partial x^a} \quad [s_1, s_2]_A = [d_A, s_1], s_2]_{\Gamma \pi A}$$

$$(\#s)(f) = [d_A, s](f)$$

THM, (Vanantvoob '97, Kostsevich '95) Let $A \rightarrow \pi$ be a VB.

There is 1-1 correspondence



Application: A morphism $\mathcal{E}: A_1 \rightarrow A_2$ between Lie algebroid is ...

• Algebroid as Schouten algebra $(A_1, Q_1), (A_2, Q_2)$ - algebras?

Schouten bracket on multi-vector fields

M , $X, Y \in \mathcal{X}(M) \Rightarrow$ the Lie bracket $[X, Y] \in \mathcal{X}(M)$

The bracket $[,]$ can be extended to $\Gamma(\wedge^* TM)$
(multi-vector fields)

by means of

$$[X, Y \wedge Z]_{SN} = [X, Y]_{SN} \wedge Z + (-1)^{(\tilde{X}+1)} Y \wedge [X, Z]_{SN}$$

($\Gamma(\wedge^* TM), [,]_{SN}$) - graded Lie algebra

In a similar construction works for Lie algebroids:

$$\cdot \Gamma(\wedge^p A) =$$

$$\bullet \cdot [s, f]_A := (\#s)(f) \quad , \quad s \in \Gamma(A) \quad , \quad f \in C^\infty(M)$$

$\therefore [s, \cdot]_A$ is a derivation of weight p of the exterior multiplication on $\Gamma(\wedge^p A)$
 $s \in \Gamma(\wedge^{p+1} A)$ $p = \text{weight of } [s, \cdot]_A$

In supergeometry language:

$$\pi A^* = (M, \mathcal{O}_{\pi A^*}) \quad , \quad \mathcal{O}_{\pi A^*}(U) = \Gamma_U(\wedge^p A) \quad , \quad A = (A^*)^*$$

i.e. πA^* is a Schouten manifold,

Double structures

● Double vector bundles (Pradines '60)

(a) A VB $E \rightarrow M$ gives diagrams

$$\begin{array}{ccc} TE & \longrightarrow & TM \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}, \quad \begin{array}{ccc} T^*E & \longrightarrow & E^* \\ \downarrow & & \downarrow \\ E & \longrightarrow & M \end{array}$$

(b) A, B, C - vector bundles over M

$$\Rightarrow D = A \times_M C \times_M B \rightarrow B$$

$$\begin{array}{ccc} & \downarrow \pi & \downarrow \pi \\ & A & \longrightarrow & M \end{array}$$

The core:
Any DVB defines the core VB $C \rightarrow M$,
 $C := \ker p_B^D \cap \ker p_A^D$.

(b) $D \xrightarrow{p_B^D} B$ two vector bundles (over A to over B)
 $\downarrow p_A^D$
 A
How to express the compatibility condition?

A VB $E \rightarrow M$ gives action of reals (\mathbb{R}, \cdot) :

$$h : \mathbb{R} \times E \rightarrow E, (t, e) \mapsto t \cdot e, t \in \mathbb{R}, e \in E.$$

Thm (2006, J. Guobinski, P.R.) E - manifold

VB structures
on E

1-1.
 \longleftrightarrow

Homogeneity structures
Actions of the monoid
 (\mathbb{R}, \cdot) on the manifold E
s.t.
...

\Rightarrow Compatibility condition for $D \rightarrow B$
 \downarrow
 A

$$h_t^A \circ h_s^B = h_s^B \circ h_t^A$$

Compare with the original definition (Mackenzie):

e.g.
$$\begin{array}{ccc}
 & D & \xrightarrow{X_A} & D \\
 +_A & \downarrow & & \downarrow \\
 & B & \xrightarrow{X_B} & B
 \end{array}
 \quad \text{(one of conditions)}$$

Local description of DVBS

$$\begin{array}{ccc}
 & D & \xrightarrow{P_B^D} & B \\
 P_A^D & \downarrow & & \downarrow \\
 & A & \xrightarrow{\quad} & M \\
 & & & (x^a)
 \end{array}$$

$$\Rightarrow C := \ker P_A^D \cap \ker P_B^D \text{ is a VB over } M$$

$$\begin{array}{l}
 A: (x^a, y^i) \\
 B: (x^a, Y^j)
 \end{array}
 \quad D: (x^a, y^i, Y^j, z^m) \text{, s.t. } C: (x^a, z^m|_C)$$

$$D: \begin{cases} \underline{y}^i = Q_j^i(x) \underline{y}^j & \text{, } \underline{Y}^I = Q_J^I(x) Y^J \\ \underline{z}^m = Q_p^m(x) \underline{z}^p + Q_{ij}^m(x) \underline{y}^i Y^j \end{cases}$$

bi-weights

$$\Rightarrow \boxed{\text{DVBS}} = \boxed{\text{graded manifolds, s.t.} \\ \text{admissible weights over} \\ 00, 10, 01, 11 \in \mathbb{Z} \times \mathbb{Z}}$$

Surprising natural duality

$$\begin{array}{ccc}
 D \rightarrow B & \Rightarrow & D_A^* \rightarrow C^* \\
 \downarrow & & \downarrow \\
 A \rightarrow M & & A \rightarrow M
 \end{array}
 \quad , \quad
 \begin{array}{ccc}
 D_B^* \rightarrow B & & \\
 \downarrow & & \downarrow \\
 C^* \rightarrow M & & M
 \end{array}$$

Thm. D_A^* is a DVBS

K. Konieczna and P. Urbański. Double vector bundles and duality (1997),

K. C. H. Mackenzie, On symplectic double groupoids and the duality of Poisson groupoids (1998)

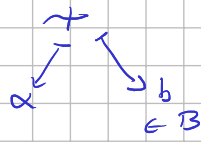
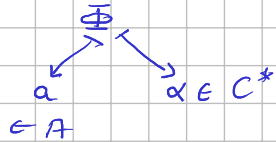
THM The VBs $D_A^* \rightarrow C^*$ and $D_B^* \rightarrow C^*$ are in

natural duality given by

$$\langle \Phi, \Psi \rangle := \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B$$

$$(\Phi, \Psi) \in D_A^* \times_{C^*} D_B^*,$$

where



and d is ANY element of D

$$\text{so } d \mapsto (a, b) \in A \times B$$

$$D \rightarrow A \times B.$$

DOUBLE LIE ALGEBROIDS

Double Lie algebroids as infinitesimal objects for double Lie groupoids

K. C. H. Mackenzie, *Double Lie algebroids and second-order geometry, I* (1992)

$S \rightrightarrows H$ two groupoid structures s.t.
 $\downarrow \downarrow$
 V the structure maps of the groupoid $S \rightrightarrows H$
 (source, target, unit, inverse) are
 groupoid morphisms s.t. second
 groupoid structure, e.g.

$$s_H : \begin{array}{ccc} S & \xrightarrow{\quad} & H \\ \downarrow & & \downarrow \\ V & & M \end{array}$$

\Rightarrow $\left. \begin{array}{l} \text{Lie functor} \\ \text{(applied twice)} \end{array} \right\}$ extremely complicated definition
 of a double Lie algebroid.

Example. Let A be a Lie algebroid. Then

(a) $\begin{array}{ccc} TA & \longrightarrow & TM \\ \downarrow & & \\ A & & \end{array}$ is a dbl. Lie algebroid

(b) $\begin{array}{ccc} T^*A \cong T^*A^* & \longrightarrow & A^* \\ \downarrow & & \downarrow \\ A & \longrightarrow & M \end{array}$ is also a DLA.

Compatibility condition for DLA

It is not enough to assume that the structure maps $\circ \perp$

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \\ A & & \end{array}$$

$D \rightarrow B$ are LA-morphisms u, v , to second algebroid structure.

III $\begin{array}{ccc} D^*A & & D^*B \\ & \searrow & \swarrow \\ & K & \end{array}$ form a DIALGEBROID.

On the other hand, Voronov's picture of Lie algebroids suggest a simple condition.

Th. Voronov, *Q-Manifolds and Mackenzie Theory* (2012)

$$\begin{array}{ccc}
 D \rightarrow B & & \pi_A D \rightarrow \pi B \\
 \downarrow & \Rightarrow & \downarrow \\
 A \rightarrow M & & A \rightarrow M
 \end{array}
 \quad , \quad
 \begin{array}{ccc}
 \pi_B D \rightarrow B & & \\
 \downarrow & & \downarrow \\
 \pi A \rightarrow M & &
 \end{array}$$

Voronov's picture $\Rightarrow Q_A \in \mathcal{X}(\pi_A D)$, $Q_B \in \mathcal{X}(\pi_B D)$

(homological vector fields corresponding to

Lie algebroids $D \downarrow A$ and $D \downarrow B$, resp.)

$\kappa(Q_A) = (1, 0)$, $\kappa(Q_B) = (0, 1)$, Q_A, Q_B are defined on different bundles BUT we have:

LEM. Let $E \rightarrow M$ be a VB. Then

$$\mathcal{X}_0(E) \cong \mathcal{X}_0(\pi E),$$

Exercise $M: (x^a)$, $TM: (x^a, \dot{x}^a)$, show that the formula

$X = \dot{x}^a \frac{\partial}{\partial x^a}$ does not define a vector field on TM .

\Rightarrow The vector fields Q_A, Q_B admit parity reversal, i.e.

$$\hat{Q}_A \in \mathcal{X}(\pi_B \pi_A D) \quad \& \quad \hat{Q}_B \in \mathcal{X}(\pi_A \pi_B D)$$

$$\hookrightarrow \pi_B \pi_A D \cong \pi_A \pi_B D.$$

$$\Rightarrow \hat{Q} = \hat{Q}_A + \hat{Q}_B \quad \text{is a well-defined v.f. on } \pi^2 D = \pi_A \pi_B D$$

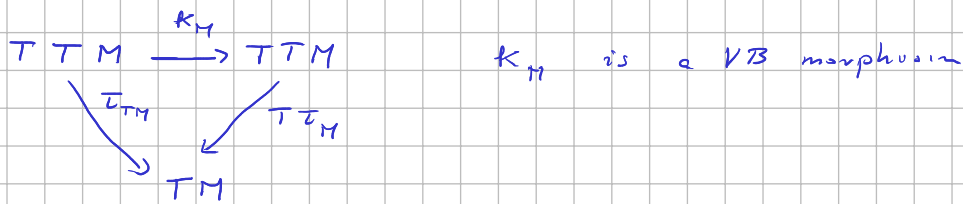
Def. A double Lie algebroid is a dbl vector bundle

equipped with old homological v.f. $Q = Q_1 + Q_2$

$$\text{i.e. } [Q, Q] = 2Q \circ Q = 0.$$

weights $(1,1)$ $(1,0)$

Tubnyjev's Kirillov-Nechevve isomorphisms



What is the dual of κ_M ?

(1) $T^*TM \xleftarrow{\alpha_M} TT^*M$ (Tubnyjev isomorphism)



The natural pairing

$$\langle \cdot, \cdot \rangle_E : E^* \times E \rightarrow \mathbb{R}$$

can be lifted to the pairing

$$\langle \cdot, \cdot \rangle_{TE} : TE^* \times TE \rightarrow \mathbb{R}$$

Here we take $E = TM$.

(2) ω_M - canonical symplectic form on T^*M

$$\Rightarrow \beta_M = \tilde{\omega}_M : TS \xrightarrow{\cong} T^*S, \quad S = T^*M$$

$$\begin{array}{ccc}
 & \xrightarrow{\beta_M \circ \alpha_M^{-1}} & \\
 T^*TM & \xleftarrow{\alpha_M} TT^*M \xrightarrow{\beta_M} & T^*T^*M
 \end{array}$$

(Tubnyjev's tuple)

The isomorphism $\beta_M \circ \alpha_M^{-1} : T^*TM \rightarrow T^*T^*M$

has a generalization to a DVVB isomorphism

$$T^*E \rightarrow T^*E^* \quad \text{for any VB } E \rightarrow M.$$

It plays a fundamental role in many problems.

(J.P. Dufour 1991,

KCH Mackenzie, P.Xu „Lie bialgebroids and Poisson groupoids“ (1994))