Talk 2 ① de Rham differential and supergeometry (Vaintrob's theorem)

M, $\Omega(M) = \bigoplus_{k\geq 0} \Omega^k(M)$, de Rham differential $d: \Omega(M) \to \Omega(M)$

has a nice interpretation: d is a vector field on a supermanifold ΠTM, $d \in \mathfrak{X}(\Pi TM)$.

(1) Supermanifolds.

$\boxed{\text{Def. A supermanifold } \mathcal{M},\ \dim \mathcal{M} = (p|q) \text{ is a } \mathbb{Z}_2\text{-graded ringed space } \mathcal{M} = (M, \mathcal{O}_{\mathcal{M}}) \text{ locally isomorphic to a superdomain in } \mathbb{R}^{p|q}.}$

top. space

sheaf of super-commutative algebras

- superdomain $= (\mathcal{U}, C^\infty_{p|q})$,

  where $\mathcal{U} \subseteq \mathbb{R}^p$ (open subset), $C^\infty_{p|q}(V) = \underbrace{C^\infty(V)}_{\text{smooth functions on } V} \otimes \underbrace{\Lambda^\bullet(\xi^1, \dots, \xi^q)}_{\text{Grassmann algebra } (\xi^i\xi^j = -\xi^j\xi^i)}$, $V \subset \mathcal{U}$

  Assume: elt. of $C^\infty(V)$ are even, ξ^i - odd.

Informally, one can think of a supermanifold as a "manifold" with both commuting & anti-commuting coordinates.

Examples A VB $E \to M$ gives a supermanifold ΠE

$$E: (x^a, y^i), \quad \begin{cases} \underline{x}^a = T^a(x) \\ \underline{y}^i = y^j T^i_j(x) \end{cases} \implies \Pi E: (x^a, \underset{\text{odd}}{\xi^i}), \quad \begin{cases} \underline{x}^a = T^a(x) \\ \underline{\xi}^i = \xi^j T^i_j(x) \end{cases}$$

Formally, $\Pi E = (M, \mathcal{O}_{\Pi E})$, $\underset{\{\mathcal{U} \subset M\}}{\mathcal{O}_{\Pi E}(\mathcal{U})} = \Gamma_{\mathcal{U}}(\Lambda^\bullet E^*)$

$$\cong C^\infty(\mathcal{U})[\xi^1, \dots, \xi^q]$$

Claim. $\Pi TM = (M, \Omega_M)$

& de Rham differential is a derivation of Ω_M,

$\implies d \in \mathfrak{X}(\Pi TM)$.

Example. $M \simeq \mathbb{C}P^1$, $\dim \mathcal{M} = (1|2)$

$$(x, \xi_1, \xi_2) \mapsto \left(y = \frac{1}{x} - \frac{1}{x^3}\xi^1\xi_2, \quad \eta_j = x^{-2}\xi_j\right), \quad j = 1, 2$$

Weinstein : Lie algebroids as „generalized tangent bundle."

$\Rightarrow$ „What can be done with the tangent bundle can be done for algebroids, as well."

- Algebroid $= (\Pi E, Q)$, $Q \in \mathfrak{X}_1(\Pi E)$.

$\Omega^\bullet(A) = \Gamma(\wedge A^*)$, $d_A : \Omega(A) \circlearrowleft$ characterized by

$$\begin{cases} (d_A f)(s) = \#(s)(f) , & f \in C^\infty(M) \ (0\text{-form}) \\ & s \in \Gamma(E) \\ (d_A \alpha)(s_1, s_2) = (\# s_1)(\alpha(s_2)) - (\# s_2)(\alpha(s_1)) - \alpha([s_1, s_2]_A) . \end{cases}$$

$\Rightarrow$ d_A determines the Lie algebroid structure, and

d_A is a derivation of $\Omega(A)$ $\Rightarrow$ $\boxed{d_A \in \mathfrak{X}_1(\Pi A)}$

$$\boxed{\begin{array}{l} A : (x^a, y^i) \\ \Pi A : (x^a, \xi^i) \\ [e_i, e_j] = Q^k_{ij} e_k \\ \# e_i = Q^a_i \partial_{x^a} \end{array}} \Rightarrow d_A = \xi^i Q^a_i(x) \frac{\partial}{\partial x^a} - \frac{1}{2} Q^k_{ij} \xi^i \xi^j \frac{\partial}{\partial \xi^k}$$

$(\mathrm{Rmk}: \Gamma(E) \simeq \mathfrak{X}_{-1}(\Pi E)$,

$[s_1, s_2]_A = [[d_A, s_1], s_2]_{T\Pi A}$

$(\# s)(f) = [d_A, s](f))$

THM. (Vaintrob '97, Kontsevich '95) Let $A \to M$ be a VB.

There is 1-1 correspondence

$\boxed{\text{Lie algebroid structures on } A \to M} \longleftrightarrow \boxed{\text{Homological vector fields } Q \in \mathfrak{X}_1(\Pi E)}$

Application : A morphism $\varphi : A_1 \to A_2$ between Lie algebroid is ... $(A_1, Q_1), (A_2, Q_2)$ – algebras ?

- Algebroid as Schouten algebra

Schouten bracket on multi-vector fields

M, $X, Y \in \mathfrak{X}(M) \Rightarrow$ the Lie bracket $[X, Y] \in \mathfrak{X}(M)$

The bracket $[\ ,\ ]$ can be extended to $\Gamma(\wedge^\bullet TM)$
(multi-vector fields)

by means of

$$[X, Y \wedge Z]_{SN} = [X, Y]_{SN} \wedge Z + (-1)^{(\tilde{X}+1)\tilde{Y}} Y \wedge [X, Z]_{SN}$$

$(\Gamma(\wedge^\bullet TM), [\ ,\ ]_{SN})$ – graded Lie algebra

In a similar construction works for Lie algebroids :

- $\Gamma(\Lambda^{\bullet} A) =$

- - $[s, f]_A := (\# s)(f)$, $s \in \Gamma(A)$, $f \in C^{\infty}(M)$

- - - $[s, \cdot]$ is a derivation of weight p of the exterior multiplication on $\Gamma(\Lambda^{\bullet} A)$
    $s \in \Gamma(\Lambda^{p+1} A)$

    p = weight of $[a, \cdot]$

In supergeometry language:

$$\Pi A^* = (M, \mathcal{O}_{\Pi A^*}), \quad \mathcal{O}_{\Pi A^*}(U) = \Gamma_U(\Lambda^{\bullet} A).$$

$A = (A^*)^*$.

i.e. ΠA^* is a Schouten mnfld,

Double structures

- Double vector bundles (Pradines '60)

(a) A VB $E \to M$ gives diagrams

$$\begin{tikzcd} TE \arrow[r] \arrow[d] & TM \arrow[d] \\ E \arrow[r] & M \end{tikzcd} \quad , \quad \begin{tikzcd} T^*E \arrow[r] \arrow[d] & E^* \arrow[d] \\ E \arrow[r] & M \end{tikzcd}$$

(b) A, B, C — vector bundles over M

$$\Rightarrow \quad \begin{tikzcd} D = A \times_M C \times_M B \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

The core: Any DVB defines the core VB $C \to M$,

$C := \ker p_B^D \cap \ker p_A^D$.

(b) $\begin{tikzcd} D \arrow[r, "p_B^D"] \arrow[d, "p_A^D"'] & B \\ A \end{tikzcd}$ two vector bundles (over A & over B)

How to express the compatibility condition?

A VB $E \to M$ gives action of reals $(\mathbb{R}, \cdot)$:

$$h : \mathbb{R} \times E \to E, \quad (t, e) \mapsto t \cdot e, \quad t \in \mathbb{R}, \; e \in E.$$

Thm (2006, J. Grabowski, M.R.) E – manifold

$$\boxed{\text{VB structures on } E} \overset{1-1}{\longleftrightarrow} \boxed{\text{Actions of the monoid } (\mathbb{R}, \cdot) \text{ on the manifold } E \text{ s.t. } \dots}$$

Homogeneity structures

$\Rightarrow$ Compatibility condition for $\begin{tikzcd} D \arrow[r] \arrow[d] & B \\ A \end{tikzcd}$ is

$$\boxed{h_t^A \circ h_s^B = h_s^B \circ h_t^A}$$

Compare with the original definition (Mackenzie):

e.g. $+_A$: $\begin{matrix} D \times_A D \\ \downarrow \\ B \times_M B \end{matrix} \longrightarrow \begin{matrix} D \\ \downarrow \\ D \end{matrix}$ (one of conditions)

Local description of DVBs

$$\begin{tikzcd} D \arrow[r, "p^D_B"] \arrow[d, "p^D_A"'] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$

(x^a)

$\Rightarrow$ $C := \ker \underset{\overset{\shortparallel}{h^A_0}}{p^D_A} \cap \ker \underset{\overset{\shortparallel}{h^B_0}}{p^D_B}$ is a VB over M

$A : (x^a, y^i)$, $B : (x^a, Y^I)$, $D : (x^a, y^i, Y^I, z^\mu)$, s.t. $C : (x^a, z^\mu|_C)$

$$D : \begin{cases} \underset{10}{\underline{y^i}} = Q^i_j(x)\, \underset{10}{y^j}, \quad \underset{01}{\underline{Y^I}} = Q^I_J(x)\, Y^J \\ \underset{11}{\underline{z^\mu}} = Q^\mu_\nu(x)\, \underset{11}{z^\nu} + Q^\mu_{ij}(x)\, \underset{10}{y^i}\, \underset{01}{Y^J} \end{cases}$$

bi-weights

$\Rightarrow$ $\boxed{\text{DVBs}}$ = $\boxed{\text{graded manifolds, s.t. admissible weights are } 00, 10, 01, 11 \in \mathbb{Z} \times \mathbb{Z}}$

Surprising natural duality

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \Rightarrow \begin{tikzcd} D^*_A \arrow[r] \arrow[d] & C^* \arrow[d] \\ A \arrow[r] & M \end{tikzcd}, \quad \begin{tikzcd} D^*_B \arrow[r] \arrow[d] & B \arrow[d] \\ C^* \arrow[r] & M \end{tikzcd}$$

Thm. D^*_A is a DVB.

K. Konieczna and P. Urbański. Double vector bundles and duality (1997),

K. C. H. Mackenzie, On symplectic double groupoids and the duality of Poisson groupoids (1998)

THM The VBs $D_A^* \to C^*$ and $D_B^* \to C^*$ are in natural duality given by

$$\langle \Phi , \Psi \rangle := \langle \Phi , d \rangle_A - \langle \Psi , d \rangle_B$$

$(\Phi , \Psi) \in D_A^* \times_{C^*} D_B^*$,

where

$$\begin{tikzcd} & \Phi \arrow[dl] \arrow[dr] & \\ a \in A & & \alpha \in C^* \end{tikzcd} \qquad \begin{tikzcd} & \Psi \arrow[dl] \arrow[dr] & \\ \alpha & & b \in B \end{tikzcd}$$

and d is ANY element of D s.t. $d \longmapsto (a, b) \in A \times B$

$D \longrightarrow A \times B$.

DOUBLE LIE ALGEBROIDS

Double Lie algebroids as infinitesimal objects for double Lie groupoids

K. C. H. Mackenzie, Double Lie algebroids and second-order geometry, I (1992)

$$\begin{tikzcd} S \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & H \\ V & \end{tikzcd}$$

two groupoid structures s.t. the structure maps of the groupoid $S \rightrightarrows H$ (source, target, unit, inverse) are groupoid morphisms w.r.t. second groupoid structure, e.g.

$$s_H : \begin{tikzcd} S \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & H \arrow[d, shift left] \arrow[d, shift right] \\ V & M \end{tikzcd}$$

$\Rightarrow$ $\left\{\begin{array}{l}\text{Lie functor} \\ \text{applied twice}\end{array}\right\}$ extremely complicated definition of a double Lie algebroid.

Example. Let A be a Lie algebroid. Then

(a)
$$\begin{tikzcd} TA \arrow[r] \arrow[d] & TM \\ A & \end{tikzcd}$$
is a dbl. Lie algebroid

(b)
$$\begin{tikzcd} T^*A \simeq T^*A^* \arrow[r] \arrow[d] & A^* \arrow[d] \\ A \arrow[r] & M \end{tikzcd}$$
is also a DLA.

Compatibility condition for DLA

It is not enough to assume that the structure maps of

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \\ A & \end{tikzcd}$$

$D \to B$ are LA-morphism w.r. to second algebroid structure.

III
$$\begin{tikzcd} D^*A \arrow[dr] & & D^*B \arrow[dl] \\ & K & \end{tikzcd}$$
form a BIALGEBROID.

On the other hand, Voronov's picture of Lie algebroids suggest a simple condition.

·Th. Voronov, Q-Manifolds and Mackenzie Theory (2012)

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} \Rightarrow \begin{tikzcd} \Pi_A D \arrow[r] \arrow[d] & \Pi B \arrow[d] \\ A \arrow[r] & M \end{tikzcd} , \begin{tikzcd} \Pi_B D \arrow[r] \arrow[d] & B \arrow[d] \\ \Pi A \arrow[r] & M \end{tikzcd}$$

Voronov's picture $\Rightarrow$ $Q_A \in \mathfrak{X}(\Pi_A D)$, $Q_B \in \mathfrak{X}(\Pi_B D)$

(homological vector fields corresponding to Lie algebroids $\begin{smallmatrix} D \\ \downarrow \\ A \end{smallmatrix}$ and $\begin{smallmatrix} D \\ \downarrow \\ B \end{smallmatrix}$, resp.)

$w(Q_A) = (1,0)$, $w(Q_B) = (0,1)$,

Q_A, Q_B are defined on different manifolds BUT we have:

THM. Let $E \to M$ be a VB. Then

$$\mathfrak{X}_0(E) \simeq \mathfrak{X}_0(\Pi E).$$

Exercise $M : (x^a)$, $TM : (x^a, \dot{x}^a)$. Show that the formula $X = \dot{x}^a \partial_{x^a}$ does not define a vector field on TM.

$\Rightarrow$ The vector fields Q_A, Q_B admit parity reversion, i.e.

$$\hat{Q}_A \in \mathfrak{X}(\Pi_B \Pi_A D) \quad \& \quad \hat{Q}_B \in \mathfrak{X}(\Pi_A \Pi_B D)$$

& $\Pi_B \Pi_A D \simeq \Pi_A \Pi_B D$.

$\Rightarrow$ $\hat{Q} = \hat{Q}_A + \hat{Q}_B$ is a well-defined v. field

on $\Pi^2 D = \Pi_A D_B$

Def. A double Lie algebroid is a Dbl vector bundle equipped with odd homological v. f. $Q = Q_1 + Q_2$, weights $(0,1)$ $(1,0)$.

i.e. $[Q, Q] = 2\, Q \circ Q = 0$.

Tulczyjew & Xu – Mackenzie isomorphisms

$$\begin{tikzcd}
TTM \arrow[rr, "\kappa_M"] \arrow[dr, "\tau_{TM}"'] & & TTM \arrow[dl, "T\tau_M"] \\
& TM &
\end{tikzcd}$$

κ_M is a VB morphism

What is the dual of κ_M ?

(1)

$$\begin{tikzcd}
T^*TM \arrow[dr, "\tau^*_{TM}"'] & & TT^*M \arrow[ll, "\alpha_M"'] \arrow[dl, "T\tau^*_M"] \\
& TM &
\end{tikzcd}$$

(Tulczyjew isomorphisms)

The natural pairing

$$\langle \cdot , \cdot \rangle_E : E^* \times_M E \to \mathbb{R}$$

can be lifted to the pairing

$$\langle \cdot , \cdot \rangle_{TE} : TE^* \times_{TM} TE \to \mathbb{R}$$

Here we take $E = TM$.

(2) ω_M - canonical symplectic form on T^*M

$$\Rightarrow \quad \beta_M = \tilde{\omega}_M : TS \xrightarrow{\simeq} T^*S , \quad S = T^*M$$

$$\begin{tikzcd}
T^*TM \arrow[rrrr, dashed, bend left, "\beta_M \circ \alpha_M^{-1}"] & & TT^*M \arrow[ll, "\alpha_M"'] \arrow[rr, "\beta_M"] & & T^*T^*M
\end{tikzcd}$$

(Tulczyjew triple)

The isomorphism $\beta_M \circ \alpha_M^{-1} : T^*TM \longrightarrow T^*T^*M$

has a generalization to a DVB isomorphism

$$T^*E \longrightarrow T^*E^* \qquad \text{for any VB } E \to M.$$

It plays a fundamental role in many problems.

(J.P. Dufour 1991,
K.CH Mackenzie, P. Xu „Lie bialgebroids and Poisson groupoids" (1994))